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Being Human: Endogenous Growth, Pollution and Natural Resources under Time-Inconsistent Preferences

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Being Human: Endogenous Growth, Pollution and Natural Resources under Time-Inconsistent Preferences

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Abstract

How does present bias shape economic growth, the creation of ideas, natural resource extraction, and pollution? We develop an endogenous growth model with exhaustible resources and pollution, where households have tendencies to procrastinate. Relative to exponential discounting, which is time-consistent, outcomes depend on an EIS threshold. If the EIS>1, present bias raises growth, shifts labor toward R&D, speeds technological innovation, and slows resource extraction; if EIS<1, economic growth falls, but effects on R&D, innovation, extraction, and pollution are theoretically ambiguous. These results highlight the crucial role of time-inconsistent behavior in shaping both economic development and environmental dynamics.

JEL Classification: O30; O44; E71; Q30, Q50; C60

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1 Introduction

Macroeconomic models widely assume that agents have time-consistent preferences, meaning their future actions will align with the plans they make today. This paper relaxes this standard assumption by introducing a general discount function to model present-biased preferences within an endogenous growth framework featuring exhaustible natural resources and pollution. Specifically, we study naive

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present-biased households—who are unaware of their future impatience—and contrast the resulting equilibrium with the standard exponential, time-consistent benchmark.

This raises several critical questions: Would agents with present bias, through a general equilibrium effect, induce the resource extraction firm to exploit the resource more aggressively than their time-consistent counterparts? Which economy would innovate and create more ideas? Would an economy populated by time-inconsistent agents grow faster or slower than one with time-consistent agents? Which economy would generate more pollution? To address these, we must understand the underlying mechanisms: What is the impact of present bias on individual consumption patterns, and how does naiveté affect these choices? A priori, one might expect that rational, time-consistent agents would achieve higher economic growth, create more ideas, extract resources more sustainably, and pollute less. However, we show that the situation is not so straightforward.

We provide closed-form answers to these questions, and our findings may be contrary to popular belief. We observe that when individuals have tendencies to procrastinate, i.e., human behaviors according to Thaler (2015), an economy with present bias can lead to higher growth rates if the constant elasticity of intertemporal substitution (EIS) is higher than one. In this scenario, the economy allocates more labor toward R&D, accelerates idea creation, and induces less aggressive resource extraction —despite enjoying faster economic growth. When the EIS is below one, present bias lowers growth. While its effects on R&D, innovation, and extraction are theoretically ambiguous, our quantitative section shows that the economy populated by time-inconsistent households allocates less labor to R&D, creates ideas at a slower pace, and extracts the resource more aggressively. We derive the theoretical conditions that lead to the mentioned outcomes. With logarithmic utility (EIS $= 1$), the two economies become observationally equivalent: present bias is irrelevant for equilibrium allocations and dynamics. These results are obtained under the observational equivalence principle (Strulik, 2015; Cabo et al., 2015, 2020a).

A key insight of our work is recognizing the fundamental importance of non-constant discounting, a concept that drives many of our findings —and a priori not all of them intuitive—. By incorporating a self-control component into our model, we address an important issue in behavioral macroeconomics — namely, the problem of willpower and the planner-doer.[1] Thus, modeling time preferences in a general setting allows us to explore their implications for the development trajectories of different economies.

Historically, the interaction and compatibility of exhaustible resources and economic growth was first rigorously analyzed in the 1970s by seminal works such as Solow (1974), Stiglitz (1974), and Dasgupta and Heal (1974). These works were a response to the book *The Limits to Growth* published by

[1]The planner–doer model is a framework that illustrates the internal conflict within an individual between long-term planning (the "planner") and short-term desires (the "doer"). Willpower refers to the individual's ability to exercise self-control to align their actions with long-term goals, resisting short-term temptations.

ecologists Meadows et al. (1972), where a pessimistic Malthusian view was presented. As a matter of course, these economists mentioned above asked: under what technological conditions can we avoid a decline in per capita consumption in the long run, given the inevitable depletion of non-renewable resources? They proposed that this decline could be counterbalanced by three key mechanisms: *i)* input substitution, *ii)* increasing returns to scale, and *iii)* resource augmenting technical progress.[2]

However, the previous authors worked within an exogenous growth framework. One of the first attempts to bridge this gap was made by Suzuki (1976), who introduced endogenous technical progress together with non-renewable resources and R&D activities that absorb part of the output, which ultimately drive technical progress. Building on this, Chiarella (1980) further endogenized the consumption-saving decision, though both studies considered market solutions without monopolistic competition or externalities.

A significant step forward was taken by Judd (1985), who proposed a model of expanding product variety. In the same decade, a set of papers changed the macroeconomics research and gave a huge momentum to the new theory of endogenous growth. The pioneering work of Romer (1986), Lucas (1988), and Rebelo (1991) built on foundational ideas from Arrow (1962),[3] Sheshinski (1967) and Uzawa (1965). Subsequent growth theories leveraging R&D incentives and imperfect competition emerged with contributions from Romer (1987, 1990), Aghion and Howitt (1992) and Grossman and Helpman (1991)[Chapters 3-4], where the R&D activity was the main driver of the creation of new ideas (technological advances). The incentives to invest in this sector were the future monopolistic reward of a given blueprint. In our paper, we follow the expanding variety approach as in Romer (1990).[4] In Romer's framework, firms invest in R&D to develop new products, and a patent system protects their innovation. This monopoly power grants higher profits, which in turn incentivizes further R&D investment (see Romer (1987, 1990) or Acemoglu (2009)[ch. 13].).

Using a different framework, a Schumpeterian literature studies resource-growth interactions: see Suphaphiphat et al. (2015) for endogenous growth with renewable resources and property-rights comparisons, Peretto and Valente (2015) for resource-population-technology interactions in a Schumpete-

[2] As Jones and Vollrath (2024) puts it, "although the world would of course be better off if natural resources were available in unlimited supply, the finite physical supplies of these resources are not a large impediment to economic growth".

[3] As written by Helpman (2004) "[t]he key argument advanced by Arrow was that information, unlike ordinary goods, can be repeatedly used by individuals and business firms without being depleted, and that individuals and business firms cannot be excluded from the use of information that becomes public. For this reason, the benefits of new knowledge are not limited to its original creators: hence the externality."

[4] As exemplified in Helpman (2004), "[f]or some versions of this model [expanding variety] the reduced-form equations - which describe the links between an economy's features and its rate of growth - are almost identical to a version of Romer's model, despite the differences in approach. For this reason, Romer's model of expanding product variety exhibits similar dynamics to the expanding quality [Schumpeterian] models of Grossman and Helpman and of Aghion and Howitt."

rian model, and Ferraro and Peretto (2018) for the role of commodity prices in shaping transitional dynamics.

A paper combining both previous ideas, endogenous growth, and exhaustible resources is the work by Scholz and Ziemes (1999), where the authors highlight that "[...] indeterminacy of equilibrium trajectories arises when the Romer (1990) model is extended for exhaustible resources. Two types of inefficiencies are responsible for this result: inefficiencies owing to (i) monopolistic behavior and (ii) information spillovers." This paper is very close to ours on the production side of the economy.

However, in all these previous works, it is assumed that agents are time-consistent (they discount the future exponentially or geometrically). The Discounted Utility (DU) Model introduced by Samuelson (1937) featured decision-makers that are characterized by having a constant discount rate, and time preferences are time-consistent. This is the well-known exponential discounting problem. Nevertheless, regarding the study of biases in intertemporal decision processes, and as clearly expressed in Myerson et al. (2001), "[d]iscounting is a pervasive phenomenon in decision making by humans and nonhuman animals. The results of a large number of experiments using delayed rewards have shown that the subjective value of a delayed reward is less than the value of an immediate reward of the same nominal amount (e.g., Green, Fry, & Myerson, 1994; Kirby, 1997; Mazur, 1987; Myerson & Green, 1995; Rachlin, 1989). More specifically, the value of a reward has been shown to decrease as a function of delay, and this phenomenon is termed temporal discounting."

The idea of discounting and self-control can be traced back at least to the work published by the Scottish philosopher Adam Smith (2010 [1759]).[5] Additionally, how human beings compare current and future events has also been the subject of extensive research in psychology and economics since the nineteenth century. Pioneering studies of intertemporal decisions can go back to the work by Rae (1834), Senior (1836) or Jevons (1879 [1871]), where the last author highlighted, "[t]he intensity of present anticipated feeling must, to use a mathematical expression, be some function of the future actual feeling and of the intervening time, and it must increase as we approach the moment of realisation. The change, again, must be less rapid the farther we are from the moment, and more rapid as we come nearer to it. An event which is to happen a year hence affects us on the average about as much one day as another; but an event of importance, which is to take place three days hence, will probably

[5] Smith wrote in his first book "*The theory of moral sentiments*" that "[t]he pleasure which we are to enjoy ten years hence interests us so little in comparison with that which we may enjoy to-day, the passion which the first excites, is naturally so weak in comparison with that violent emotion which the second is apt to give occasion to, that the one could never be any balance to the other". Moreover, there is a canonical illustration of time-inconsistency in the classic literature, the episode of Ulysses and the Sirens. In Homer's *Odyssey*, Ulysses, fully aware of his future vulnerability to the seductive song of the Sirens, orders his crew to tie him to the mast so that he cannot steer his ship toward danger when he hears them singing. This "Ulysses contract" is an early, powerful example of how individuals can take preemptive measures to counteract their anticipated deviations from their intended plans, a central theme in the study of intertemporal decision-making.

affect us on each of the intervening days more acutely than the last". Fisher (1930), who used the term "impatience", argued that these time preferences showed a lack of foresight and self-control. Moreover, Ramsey (1928)[p. 543] preferred to use a discount rate equal to zero.[6] In addition, Pigou (2017 [1920]) also thought that discounting was a sign of the poor functioning of our imagination.[7] Nevertheless, despite the reluctance of previous authors, it was in the work by Samuelson (1937) when the DU model was popularized, that is, when agents discount exponentially. Strotz (1955) showed that exponential discounting with constant rates of time preferences was the only discount function that generated time consistency, and Phelps and Pollak (1968) introduced the (quasi)hyperbolic (or quasigeometric) discount functions in discrete time.

A rich empirical literature supports the notion of time-varying discount rates (see, for instance, Thaler (1981); and for comprehensive surveys, see Frederick et al. (2002), DellaVigna (2009), and Ericson and Laibson (2019)). Some authors give validity to the hypothesis that individuals are simply not aware of their future impatience (see O'Donoghue and Rabin (1999a,b, 2001), Caliendo and Aadland (2007), and Findley and Caliendo (2014)). Additionally, it is claimed that individuals invest or save too little when they discount the future hyperbolically and are subject to self-control problems (Laibson, 1997, 1998). [8]

The use of continuous-time models to study present bias has gained significant momentum in recent years (Grenadier and Wang, 2007; Cao and Werning, 2016, 2018; Acharya et al., 2023; Beshears et al., 2022; Shigeta, 2022). Notable recent contributions include Maxted (2025), who analyzes consumption-saving decisions with stochastic income, risky and illiquid assets, and costly borrowing. Maxted et al. (2025) study the effect of monetary and fiscal policy in a heterogeneous-agent model where households have present-biased time preferences and naive beliefs. These previous two papers, however, use the continuous-time counterpart of the particular $\beta - \delta$ model, and what they define as "instantaneous gratification", whereas we work with a general discount function.

Bassetto et al. (2024) study how time-inconsistent policymakers —whether due to short-term horizons or incentive misalignments— can delay the adoption of "good" policies in a discrete time setting. They propose an "organizational equilibrium" concept in which decision-makers may choose to wait and avoid addressing the issue, or replicate their predecessors' actions, shedding light on how institu-

[6] His justification was that "we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination."

[7] Pigou (2017 [1920]) wrote: "[b]ut this preference for present pleasures does not - the idea is self-contradictory - imply that a present pleasure of given magnitude is any greater than a future pleasure of the same magnitude. It implies only that our telescopic faculty is defective, and that we, therefore, see future pleasures, as it were, on a diminished scale."

[8] For instance, due to the new technologies, such as having apps on our phones that make it easier to shop online, we tend to buy more things than we would if we had to go to a physical store, keeping us away from saving for our future. This is because these apps are designed to be addictive and keep us coming back for more (Alter, 2018).

tions can evolve in the absence of commitment.[9]

Other authors argue that humans behave in a sophisticated, or time-consistent manner, as Barro (1999), Karp (2007), Karp and Tsur (2011), Tsoukis et al. (2017) or Cabo et al. (2020a). The first author of this latter group initiated the literature on non-constant discounting and economic growth by studying the logarithmic and power utility cases. He concluded that for a neoclassical growth model with log utility, exponential discounting is observationally equivalent to quasi-hyperbolic (non-constant) discounting. This is an old result in the study of hyperbolic discounting, where the behavior of sophisticated and naive agents coincides under a logarithmic utility function (Pollak, 1968). Obviously, this is precisely what we get for our time-inconsistent agent under an endogenous growth model with natural resources. Nonetheless, this equivalence does not hold when we consider elasticities of substitution different from one (Marín-Solano and Navas, 2010; De-Paz, Marín-Solano, Navas, and Roch, 2014; De-Paz, Marín-Solano, and Navas, 2013; Farzin and Wendner, 2014). This result will play a crucial role in our analysis.

Recent contributions that merge both previous fields, non-constant discounting and endogenous growth are Strulik (2015), where intermediate good firms operate under perfect competition, Cabo et al. (2015, 2016, 2020a), and Cabo et al. (2020b), where a basic AK model is used in the production side. In contrast, our framework builds on the *Expanding Variety Models* by incorporating monopolistic competition and an explicit resource extraction sector. This implies that firms can charge their markup due to their monopolistic power and the resource sector takes into account the dynamics of the resource. Strulik (2015) proves that exponential and hyperbolic discounting are observationally equivalent for log-utility and naive consumers. Here we prove a more general case for any non-constant discounting, not only hyperbolic discounting.

Furthermore, pollution in Cabo et al. (2020b) is extended to be a byproduct of the production of the final good and the extraction of the exhaustible resource. Additionally, Dugan and Trimborn (2020) study the effects of a declining social discount rate on the optimal extraction of non-renewable resources and economic growth with hyperbolic discounting. In their work, they show that resource use —extraction— is more conservative under hyperbolic discounting in the first periods. They study the Dasgupta-Heal-Solow-Stiglitz (DHSS), which is an exogenous growth model and do not consider pollution. We generalize this idea, and will obtain different results depending on the value of the EIS. We also obtain their more conservative extraction for present bias agents with an EIS lower than one,

[9] In the debate between continuous and discrete-time models with present bias, Maxted (2025) advocates for a continuous-time approach. He states that continuous-time specification of present bias "closely approximate discrete-time models with period lengths that are psychologically appropriate", where "laboratory studies find that the temporal division between "now" and "later" is less than one week. However, discrete-time consumption-saving models typically use either quarterly or annual time-steps that are inconsistent with the high frequency at which present bias generally operates".

but we get the opposite results if agents have an EIS greater than one —more aggressive extraction rate for an economy with non-constant discounting. Interestingly, non-constant discounting agents start extracting more aggressively in the first periods when the EIS is higher than one ($\sigma > 1$), which is a different behavior observed in the long run, i.e., the resource is extracted less aggressively if agents are present-biased. Since in the first periods more is extracted with a general discount, this implies that in the medium and long term there is less natural resource left to be extracted.

Combining these elements, we develop an endogenous growth model with monopolistic competition, exhaustible natural resources, and pollution under non-constant discounting. In our framework, the EIS plays a pivotal role: When the elasticity of intertemporal substitution (EIS) exceeds one, present bias leads to higher economic growth, allocates more labor toward R&D, accelerate idea creation, and—despite stronger economic dynamism—induce less aggressive resource extraction. If the EIS is below one, present bias lowers growth for sure. However, its effects on R&D, innovation, extraction, and pollution are theoretically ambiguous. In the quantitative section, we calibrate the model and find that time-inconsistent agents allocate less labor to R&D, create ideas at a slower pace, extract the resource more aggressively, and decrease pollution faster (if pollution is decreasing) or exhibit lower pollution growth rates (if pollution is increasing). With log utility (EIS $= 1$), the two economies are observationally equivalent. We derive closed-form expressions and characterize the stability of the model. The results highlight that whether present bias helps or harms development and the environment depends on which side of the EIS$= 1$ knife-edge an economy lies.

This paper is organized as follows. Section 2 describes the production side of the economy. Section 3 studies how households behave. Section 4 shows the general (market) equilibrium. The quantitative analysis is carried out in Section 5, and Section 6 concludes.

2 Production Sector

The production side of the model comprises four sectors: the resource extraction sector, the final goods sector, the intermediate goods sector, and the R&D sector.

2.1 Representative Firm of the Resource Sector

We consider the natural resource to be an asset with well-defined property rights, abstracting from the problems of common pool resources. The resource sector initially buys the existing stocks from households and maximizes its profits. Later on, these profits generated by the single firm in the resource sector will be acquired by households, who are the owners of this sector. The extracted resource is later supplied to the final-good sector at a price $p_R(s)$. The dynamics of the non-renewable resource $S(s)$ is given by

$$\frac{dS(s)}{ds} = -R(s), \tag{1}$$

which implies that the total resource extracted over time, $\int_{t_0}^{\infty} R(s)\,ds$, does not exceed the initial stock $S(t_0)$ where t_0 is the initial time at which the resource sector starts extracting the exhaustible natural resource. There are no extraction costs.[10] The discounted profit of the resource extraction firm is defined as

$$\Pi^{RS} \equiv \int_{t_0}^{\infty} e^{-\int_{t_0}^{s} r(h)dh} p_R(s)R(s)ds, \tag{2}$$

where $r(s)$ denotes the interest rate at time s and $p_R(s)$ is the price of the resource. One can see why the firm in the resource sector does not extract the whole resource today and get rich immediately, as it has to think about its future selves and leave resources underground to extract tomorrow and keep generating profits. Therefore, the equilibrium condition is

$$\gamma_{p_R} \equiv \frac{\dot{p_R}(s)}{p_R(s)} = r(s), \tag{3}$$

which is the well-known Hotelling rule, stating that the growth rate of the price of the resource is equal to the interest rate (see Online Appendix B.1).

2.2 Representative Firm of the Final-Good Sector

The final good is produced competitively at time s with the following production function:

$$Y(s) = \frac{1}{1-\beta_1-\beta_2}\left(\int_0^{N(s)} x(\nu,s)^{1-\beta_1-\beta_2} d\nu\right) L_F(s)^{\beta_1} R(s)^{\beta_2}, \tag{4}$$

where $L_F(s) \in (0,1)$ denotes the labor employed in final goods production, $R(s)$ is the amount of resource used by the final-good producer and supplied by the resource sector, which is normally also interpreted as the energy input used in this final sector (Golosov et al., 2014; Jones and Vollrath, 2024). $N(s)$ represents the number of distinct input varieties, and $x(\nu,s)$ is the amount of input of variety $\nu \in [0, N(s)]$ used at time s.[11] Moreover, the elasticities of labor and exhaustible resource used in the final production are β_1 and β_2 respectively.[12] The final good production function can be rewritten as,

[10] However, this could be interpreted as a reduced price. If there were a constant marginal cost c_0 and a price P_R^{cost}, the instantaneous profit would be $(P_R^{cost} - c_0)R(s)$. One could define this difference as the price we use.

[11] In his book *Big Ideas in Macroeconomics: A Nontechnical View*, Kartik B. Athreya (2013) writes, "[...] an industry of price-taking, profit-maximizing firms will *look as if* it had set out to solve the profit maximization of a fictitious single firm that embodies the entire economy's production capabilities. Of course, no one is doing any".

[12] This follows ideas in Acemoglu (2002), where the author writes "[n]otice that given $[N(s)]$, the production functions [...] exhibit[s] constant returns to scale. There will be aggregate increasing returns, however, when $[N(s)$ is] endogenized."

$$Y(s) = \frac{1}{1-\beta_1-\beta_2}\tilde{\mathbf{X}}(s)^{1-\beta_1-\beta_2}L_F(s)^{\beta_1}R(s)^{\beta_2}, \tag{5}$$

where

$$\tilde{\mathbf{X}}(s) \equiv \left(\int_0^{N(s)} x(\nu,s)^{\frac{\varepsilon_\beta-1}{\varepsilon_\beta}}d\nu\right)^{\frac{\varepsilon_\beta}{\varepsilon_\beta-1}},$$

and $\varepsilon_\beta \equiv \frac{1}{\beta_1+\beta_2}$ is the elasticity of substitution between different input varieties, with $\varepsilon_\beta > 1$. This formulation emphasizes the parallel with the Dixit-Stiglitz framework and shows that the production exhibits constant returns to scale, i.e., is homogeneous of degree one.

The demand for each input variety $x(\nu,s)$ is determined by maximizing the aggregate profits of the final-good producer. Intermediate monopolists supply each variety and set a price $p^x(\nu,s)$ at time s for machine ν to maximize their profits. This is normally interpreted as a "rental price" or the cost of that machine ν. The final producer also hires labor at a wage rate $w(s)$, and uses the resource $R(s)$ at price $p_R(s)$. We normalize the price of the final good to 1. The first order conditions with respect to intermediate inputs, labor, and resources are

$$x(\nu,s)^{-\beta_1-\beta_2}L_F(s)^{\beta_1}R(s)^{\beta_2} = p^x(\nu,s), \tag{6}$$

$$\beta_1\frac{Y(s)}{L_F(s)} = w(s), \text{ and} \tag{7}$$

$$\beta_2\frac{Y(s)}{R(s)} = p_R(s), \tag{8}$$

where eq. (6) is the demand for machine $\nu \in [0,N(s)]$ by the final good producer, which can be expressed in the following isoelastic form:

$$x(\nu,s) = p^x(\nu,s)^{-1/(\beta_1+\beta_2)}L_F(s)^{\beta_1/(\beta_1+\beta_2)}R(s)^{\beta_2/(\beta_1+\beta_2)}. \tag{9}$$

This expression shows that the demand for each input variety decreases with its price and increases with the amounts of labor and resource used, while it is independent of the wage, the resource price, and the total number of varieties $N(s)$. Intuitively, this demand shows directly the elasticity of the demand for different intermediate inputs ε_β, which would allow us to rewrite the previous expression as $x(\nu,s) = p^x(\nu,s)^{-\varepsilon_\beta}L_F(s)^{\beta_1\varepsilon_\beta}R(s)^{\beta_2\varepsilon_\beta}$.

2.3 Representative Firm of the Intermediate-Good Sector

The intermediate-good sector is responsible for the supply of machines or intermediate goods of type $\nu \in [0,N(s)]$ used by the final sector. Each machine of variety $\nu \in [0,N(s)]$ is assumed to fully

depreciate after use.[13] Each intermediate good is only produced by a monopolist ν who charges a price $p^x(\nu, s)$. The design for such goods is invented in the R&D sector, developed later. The present discounted value of owning the blueprint of a machine type ν is given by

$$V(\nu, s) = \int_s^{\infty} \exp\left(-\int_s^{\tau} r(\xi)d\xi\right) \pi(\nu, \tau)d\tau, \tag{10}$$

where $r(\xi)$ represents the monopolist's risk-free discount rate. The profit of monopolist ν at time s is defined as

$$\pi(\nu, s) := p^x(\nu, s)x(\nu, s) - rx(\nu, s). \tag{11}$$

In this case, we assume that one unit of machine ν can be produced at marginal cost $r > 0$ units of the final good as in Scholz and Ziemes (1999). Thus, any monopolist $\nu \in [0, N(s)]$ maximizes its profits defined in eq. (11). We also assume that the firm that discovers a new blueprint will receive a perpetual patent for this new invention, as it is normally assumed in the literature (Romer, 1990; Acemoglu, 2002). Using the demand for a machine of type ν in eq. (9) and plugging it into the monopolist profit function gives the optimal price condition:

$$p^x(\nu, s) = \frac{r}{1 - \beta_1 - \beta_2} \equiv p^x \in \mathbb{R}_+ \text{ for all } \nu \text{ and } s. \tag{12}$$

Thus, any monopolist ν charges a constant rental price which is equal to a markup over the marginal cost r. Alternatively, one can write the first order condition of the monopolist as

$$r = (1 - \beta_1 - \beta_2)x(\nu, s)^{-\beta_1-\beta_2} L_F(s)^{\beta_1} R(s)^{\beta_2}. \tag{13}$$

Further details are provided in the Online Appendix B.2.

2.4 Representative Firm of the R&D Sector

In the economics of ideas, technology or recipes are very different goods from raw materials or finished tangible goods such as a MacBook Air or a table.[14] Here we follow the "growth with knowledge spillovers" approach, where scientists and researchers (the R&D labor force, denoted by $L_{RD}(s)$) are the primary creators of new ideas. This implies a "scarce factor" used in the development and research of new intermediate goods. Therefore, economic growth under this specification will be driven by

[13] This assumption ensures that "the amount of these machines used in the past are not additional state variables" (Acemoglu, 2009, p.434), considerably simplifying the model. Thus, $p^x(\nu, s)$ can be interpreted as the cost of using this machine or a "rental price".

[14] By technology, one should understand the "formula" or knowledge that allows companies to "combine" labor and capital to produce a product. Ideas are particular goods in economics, as they are non-rival and partially excludable (Romer, 1990).

knowledge spillovers from past R&D, often described as "standing on the shoulders of giants". Accordingly, the innovation possibilities frontier is given by

$$\dot{N}(s) = \eta N(s) L_{RD}(s), \tag{14}$$

where $\eta > 0$, and $N(s)$ represents the existing stock of ideas.[15] This expression will be the key factor in understanding the source of endogenous growth since the spillover effects are linear or proportional.

We assume that the total labor force is allocated between the final-good sector and the R&D sector, with total labor normalized to 1. Therefore, the corresponding market-clearing condition is $L_{RD}(s) + L_F(s) \leq L \equiv 1$. The firm in the R&D sector maximizes the profit function $\pi^{RD} = p_{RD}(s) \cdot \eta N(s) \cdot L_{RD}(s) - w(s) L_{RD}(s)$ where households supply labor to this sector and receive a wage $w(s)$. The price this sector charges for a produced patent is $p_{RD}(s)$. The previous profit of the R&D gives the following equilibrium condition,

$$w(s) = \eta \cdot p_{RD}(s) N(s). \tag{15}$$

2.5 Pollution

Finally, we describe how pollution is generated in our model. Pollution arises from two sources: the extraction of the non-renewable resource, $R(s)$, and the production of the final good, $Y(s)$,

$$P(R(s), Y(s)) = R(s)^{\mu_1} \cdot Y(s)^{\mu_2}, \tag{16}$$

where μ_1 and μ_2 are constant over time.[16] In Cabo et al. (2020a) and Cabo et al. (2020b), pollution is positively related only to the input capital, while in our setting, we consider that both extracting a non-renewable resource and producing the final good generates pollution.

[15] In expanding variety models, η would capture the productivity/efficiency in the R&D sector (number of new ideas/number of researchers). However, here, in equilibrium, the optimum allocation of labor to research and development activities will depend on η as well, i.e., $L_{RD}(\eta)$, and that interpretation will not be that straightforward. Additionally, this equation shows the spillover effect: the more knowledge available, the more productive each researcher becomes, leading to further innovation. We should understand this creation of new ideas in a broader sense. As pointed out by Bhide (2003) in his book *The Origin and Evolution of New Business*, 72% of the ideas come from workers, 20% from ordinary people (students, professors) and only 7% come directly from the R&D. Furthermore, recent empirical papers study whether new ideas are harder to find (Bloom et al., 2020) in contrast to the scheme previously exposed. This extension is left for future research.

[16] While one might be tempted to interpret μ_1 and μ_2 as the elasticities of pollution with respect to resource extraction and final output, meaning that a 1% increase in $R(s)$ or $Y(s)$ would result in a μ_1% or μ_2% increase in pollution, this interpretation is not accurate. This is because an increase in resource extraction $R(s)$, for instance, not only directly contributes to pollution via the $R(s)^{\mu_1}$ component, but also indirectly by serving as an input to $Y(s)$, which then contributes to pollution via $Y(s)^{\mu_2}$. Thus, μ_1 primarily reflects the environmental impact intensity of extracting and initially processing the resource,

The rationale for modeling both activities contributing to pollution is straightforward. It is widely accepted that the very act of extracting a non-renewable natural resource generates pollution. At the same time, the production of the final good further adds to environmental degradation. If one were interested only in the direct impact of final production on pollution, the parameter μ_1 could be set to zero. However, in this paper, we study the more general case in which both extraction and production generate pollution.

3 Households

Households have the following constant relative risk aversion (CRRA) or constant elasticity of intertemporal substitution (EIS) instantaneous utility function,

$$U\left(c(s), P(s)\right) = \begin{cases} \frac{c(s)^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}} - \psi\frac{P(s)^{1+b}-1}{1+b} & \text{if } \sigma \in \mathbb{R}_{++} \setminus \{1\} \wedge b \neq -1, \\ \ln c(s) - \psi \ln P(s) & \text{if } \sigma = 1 \wedge b = -1, \end{cases} \tag{17}$$

where $c(s)$ is the level of consumption at time s, and $\sigma \in \mathbb{R}_{++}$ represents the constant elasticity of intertemporal substitution, EIS, and $1/\sigma$ captures the Arrow-Pratt measure of relative risk aversion (RRA). Under our isoelastic utility, EIS $\sigma < 1$ implies a stronger desire to smooth consumption, while EIS $\sigma > 1$ implies a weaker desire to smooth. Here consumption and pollution are separable (see also Stokey (1998), Acemoglu et al. (2012), and Jones and Vollrath (2024)). The parameter ψ captures the curvature of how pollution generates disutility.[17] The parameter b captures the relative weight of pollution in the utility function. Households choose how much to consume and save, holding assets in the form of both capital and bonds.The presence of $N(s)$ monopolists implies that household wealth/assets, $a(s)$, consists of $N(s)$ bonds -priced at $p_{RD}(s)$- issued by these monopolists and total demand of capital of all the $N(s)$ monopolists. In other words, total household assets are determined by the number of bonds (or patents) and the aggregate demand for the capital held by the $N(s)$ monopolists, such as

$$a(s) \equiv N(s)[p_{RD}(s) + \underbrace{k(\nu, s)}_{\equiv x(\nu,s)}]. \tag{18}$$

Besides, it is assumed that capital and bonds are perfect substitutes "as stores of value", implying that they must yield the same interest rate. Moreover, monopolists use capital stock to produce intermediate goods. Each monopolist ν issues bonds to finance the purchase of a patent. Consequently,

while μ_2 captures the pollution intensity of converting inputs into final output.

[17] For values of b lower than -1, the disutility of pollution is not that strong, and the second derivative is positive. For values $b \in (-1, 0)$ the damage is still convex. For $b = 0$, the disutility of pollution is linear. For $b > 0$, the disutility of pollution is concave, that is, the greater the levels of pollution, the greater the loss in utility. We believe this is a more realistic case. The same qualitative results hold if $\psi = 0$.

household income comes from holding bonds, $rN(s) \cdot p_{RD}(s)$; capital income, given by $rK(s)$, with aggregate capital stock defined as $K(s) \equiv N(s), k(s)$; income from the monopolistic resource sector, $p_R(s) \cdot R(s)$; and labor income, which is earned both in the final goods sector, $wL_F(s)$, and in the R&D sector, $wL_{RD}(s)$. As the labor force is normalized to one, total labor income is simply given by the wage w, since $wL_F(s) + wL_R(s) = w$. Households save in capital $K(s)$ and bonds.[18]

3.1 On Time Preferences

Time preferences are captured by the discount function $\theta(s-t) \geq 0$ that depends not only on the time when the control (consumption) is enjoyed at time s, but also on the time distance from the present t (when the decision is made). This time distance between when consumption is enjoyed and when the decision is made is defined as $j \equiv s - t$. The function $\theta(j)$ satisfies the following properties: $\theta(j) > 0$, $\dot{\theta}(j) < 0$, for all $j > 0$, and $\theta(0) = 1$.

Thus, the *instantaneous discount rate* $\rho(j) \equiv -\dot{\theta}(j)/\theta(j)$ is non-increasing with the time distance from the present, i.e., $\rho(j) > 0, \dot{\rho}(j) \leq 0$, for all $j > 0$ (Laibson, 1997; Barro, 1999). It can also be defined as the (negative) growth rate of the discount function. Intuitively, this means that the further in the future an event occurs, the "less important" it is for the present agent at time t. In the standard time distance exponential discounting popularized in Samuelson (1937), where $\theta^{Exp}(s-t) = e^{-\hat{\rho}(s-t)} = e^{-\hat{\rho}j} = \theta^{Exp}(j)$, it is easy to see that $\rho(j) = -\dot{\theta}(j)/\theta(j) = \hat{\rho} \in \mathbb{R}_{++}$ is constant, in comparison to a general discount function. Put differently, the logarithmic rate of change of the discount is constant, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}j}\left[ln\left(\theta(j)\right)\right] = \frac{\mathrm{d}}{\mathrm{d}j}\left[ln\left(e^{-\hat{\rho}j}\right)\right] = \frac{\mathrm{d}}{\mathrm{d}j}\left[-\hat{\rho}j\right] = -\hat{\rho}.$$

This ensures time-consistency in standard inter-temporal problems (Strotz, 1955). To capture more realistic behavioral patterns, however, several alternative discount functions have been proposed. For instance, Tsoukis et al. (2017), Luttmer and Mariotti (2003), and Cabo et al. (2020a) use a specification that combines "generalized hyperbolic discount function" with the exponential discount,[19]

$$\theta^{Tsouk.}(j) = (1+\delta j)^{-\varphi/\delta} e^{-\rho j}, \tag{19}$$

where $\rho > 0$, $0 < \varphi < 1$, $\delta > 0$ and $\varphi/\delta < 1$. Note that $\theta(j)$ decreases with the time distance j, $\frac{\partial}{\partial j}\left[(1+\delta j)^{-\varphi/\delta} e^{-\rho j}\right] = -(1+\delta j)^{-\varphi/\delta} e^{-\rho j}\left[\rho + \frac{\varphi}{1+\delta j}\right] < 0$ and $\frac{\partial^2}{\partial j^2}\left[(1+\delta j)^{-\varphi/\delta} e^{-\rho j}\right] > 0$. Importantly, as φ tends to 0, this discount function converges to the standard exponential discount function.

[18] See Scholz and Ziemes (1999) for a discussion of the budget constraint, and page 77 in Aghion and Howitt (2008) where the second term in (18) is defined in the same way.

[19] As explained in Luttmer and Mariotti (2003), "[i]t generalizes the hyperbolic discount function proposed by Ainslie (1975) to interpret experiments that indicate reversals over time of preferences for rewards at different horizons."

The corresponding instantaneous discount rate is $\rho(j) = \rho + \frac{\varphi}{1+\delta j}$. Rising ρ increases the discount rate at all horizons, and rising φ increases the subjective discount rate more in the short run than at long horizons.

Another specification is a "convex linear combination of two exponentials", as for instance in Karp (2007),

$$\theta^{Conv.Exp}(j) = ze^{-\rho_1 j} + (1-z)e^{-\rho_2 j}, \tag{20}$$

where $z \in (0,1)$ and $\rho_1 < \rho_2$, with a decreasing instantaneous discount rate $\rho^{Conv.Exp}(j) = \frac{z\rho_1 e^{-\rho_1 j} + (1-z)\rho_2 e^{-\rho_2 j}}{ze^{-\rho_1 j} + (1-z)e^{-\rho_2 j}}$. With this discount function, ρ_1 captures the level of impatience in the long run, and ρ_2 captures the level of impatience in the short run. Consequently, higher values of ρ_2 imply higher impatience in the short run. This function decreases from $\rho(0) = z\rho_1 + (1-z)\rho_2$ to its level of impatience in the long run $\lim_{j\to\infty} \rho^{Conv.Exp}(j) = \rho_1$, as $\rho_1 < \rho_2$.

Figure 1 illustrates several discount functions and their corresponding instantaneous discount rates, highlighting that under non-constant discounting the short run is discounted more heavily while the long run is valued relatively more than under exponential discounting.

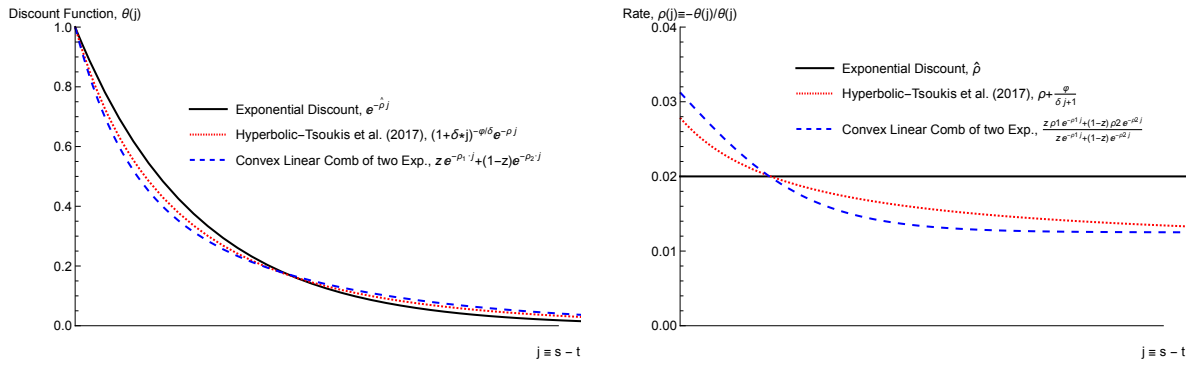


Figure 1: Discount Functions $\theta(j)$ and Instantaneous Discount Rate $\rho(j) \equiv -\dot{\theta}(j)/\theta(j)$.

In order to compare the standard exponential discounting with non-constant discounting, we follow the *observational equivalence principle*, i.e., find the discounting parameters such that the discounted infinite stream gives the same present value,[20]

$$\int_0^\infty e^{-\hat{\rho} j} dj \stackrel{!}{=} \int_0^\infty \theta(j) dj, \quad \Longleftrightarrow \quad \hat{\rho} = \left[\int_0^\infty \theta(j) dj\right]^{-1}. \tag{21}$$

As noted by Cabo et al. (2020a), "under non-constant discounting the short run is less valued and the long run more valued than under exponential discounting". This result comes from the fact that

[20] See Myerson et al. (2001), Strulik (2015), Cabo et al. (2015, 2020a), and Mañó-Cabello et al. (2021)

the instantaneous discount rate $\rho(j)$ is large at first and decreases with the time distance, jointly with the observational equivalence principle in (21).

At each time t, a naive household solves the following intertemporal optimization problem:[21]

$$\max_{\{c_t(s)\}_{s\geq 0}} \quad \int_t^{\infty} \theta(s-t)\, U\big(c_t(s), P_t(s)\big)\, \mathrm{d}s$$
$$\text{s.t.} \quad \dot{a}_t(s) = r_t(s)\, a_t(s) + w_t(s) + p_{R_t}(s)\, R_t(s) - c_t(s),$$

where $a(s)$ represents household wealth or savings, defined in eq (18), and $a_t(t) = a_t \in \mathbb{R}$ is given. We now compute the naive solution to the household optimization problem, commonly referred to as the *t-agent* problem. As pointed out by Marín-Solano and Navas (2009), the "[n]aive t-agent will solve [the] problem [...] as a standard optimal control problem, using Pontryagin's Maximum Principle" (see also Strulik (2015) and Cabo et al. (2015)). Consumption is then given by (see Appendix B.3 for its derivation),

$$c_t(t) = \left(\frac{a_t + \int_t^{\infty} [w_t(s) + p_R(s) * R(s)] \cdot \exp\left\{ -\int_t^s r_t(\xi) d\xi \right\} ds}{\int_t^{\infty} \theta(s-t)^{\sigma} \cdot \exp\left\{ -(1-\sigma) \int_t^s r_t(\xi) d\xi \right\} ds} \right). \tag{22}$$

The solution is written in feedback form. This result is similar to the one obtained in Cabo et al. (2015), but we now consider a broader income that also comes from the resource sector. Expression (22) represents the optimal consumption of households under non-constant discounting. The numerator captures the total wealth (physical and human) $TW(s)$, and the denominator represents the inverse of the marginal propensity to consume (MPC) out of total wealth (Farzin and Wendner, 2014). Thus, the consumption strategy can be seen as $c_t(t) = \frac{1}{MPC} \times TW(s)$, and as a linear Markovian strategy, i.e., consumption (control) is a linear function of wealth (state). The growth rate of consumption is given by the following modified Ramsey rule:

$$\gamma_c(\sigma, \theta) \equiv \frac{\dot{c}(t)}{c(t)} = \sigma \left[r_t - \lambda_{\theta}^{ces} \right], \tag{23}$$

where

$$\lambda_{\theta}^{ces} = \frac{\int_0^{\infty} \rho(j) \theta(j)^{\sigma} \exp\left\{ -(1-\sigma) \int_t^s r_t(\xi) d\xi \right\} dj}{\int_0^{\infty} \theta(i)^{\sigma} \exp\left\{ -(1-\sigma) \int_t^s r_t(\xi) d\xi \right\} di} \equiv \lambda_{\theta}(\sigma, r_t, t). \tag{24}$$

For simplicity, we write λ_{θ}^{ces} instead of $\lambda_{\theta}(\sigma, r_t, t)$. The term λ_{θ}^{ces} is the *effective rate of time preference* and will be constant in the BGP. It summarizes the household's overall impatience under the chosen discount function. It can be interpreted as a "weighted mean of the instantaneous discount rates $\rho(j)$" (Cabo et al., 2015), or as in Barro (1999), the "weighted average of the instantaneous rates of time preference".[22] It is worth noting that effective rates of time preference λ_{θ}^{ces} are specific to each

[21] Note that the household will be time-consistent if they have an exponential discount function, and time-inconsistent if they have another discount function

[22] Luttmer and Mariotti (2003) also call this the "weighted average of the subjective discount rates".

discount function $\theta(s-t)$. The weights are captured by $\omega_N(j)$ with $\int_0^\infty \omega_N(j)dj = 1$.[23] Thus, the effective rate of time preference can be expressed as a weighted mean $\lambda_\theta^{ces} = \int_0^\infty \rho(j)\omega_N^{ces}(j)dj$.

Moreover, when the interest rate is constant, the household problem in Cabo et al. (2015) is recovered, even if one considers pollution and income from the resource sector. Rearranging the modified Ramsey rule (23) yields a transparent and intuitive *consume-versus-save* condition:

$$\lambda_\theta^{ces} + \frac{1}{\sigma}\frac{\dot{c}(t)}{c(t)} = r_t,$$

where the left-hand side (LHS) captures the benefit derived by consumption and λ_θ^{ces} expresses the increase in utility derived by consuming now and not postponing consumption to the "next period". Moreover, the term $\frac{1}{\sigma}\frac{\dot{c}(t)}{c(t)}$ captures the household's desire to smooth consumption.[24] Furthermore, the right-hand side (RHS) term is the yield or net benefit obtained from saving. Thus, this expression shows the threshold (margin) where agents would be indifferent between consuming or saving.

If agents have logarithmic utility, then, λ_θ^{log} collapses to $\left[\int_0^\infty \theta(i)di\right]^{-1} = \hat{\rho}$, for any general discount rate $\theta(j)$, where $\hat{\rho}$ is given by the equivalent present value expression (21) (Barro, 1999; Cabo et al., 2015). Hence log utility "washes out" time-inconsistency: exponential and non-exponential discounters obey the same Euler equation.[25]

Remark 1. *Under logarithmic utility ($\sigma = 1$), the weighted average of the instantaneous discount rates collapses to the constant discount rate of the exponential discount function,*

$$\lambda^{log} = \hat{\rho}, \tag{25}$$

which is then independent of the discount function $\theta(j)$.

Proof. See Online Appendix B.4. □

Together with the observational equivalence condition, when $\sigma = 1$, this implies identical Euler equations and equilibrium outcomes for any discount function $\theta(.)$, even when agents have present bias.

[23] One can write thus

$$\lambda_\theta^{ces} = \int_0^\infty \rho(j) \underbrace{\frac{\theta(j)^\sigma \exp\left\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\right\}}{\int_0^\infty \theta(i)^\sigma \exp\left\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\right\} di}}_{\equiv \omega_N^{ces}(j)} dj.$$

[24] Along any path with $\frac{\dot{c}(t)}{c(t)} > 0$, households would prefer to increase consumption today. Since households have the desire to smooth consumption, if they know that in the future they will consume more ($\dot{c}(t)/c(t) > 0$), driven by the behavior force to smooth their consumption, they will have incentives to increase consumption today.

[25] See also Marín-Solano and Navas (2010), De-Paz, Marín-Solano, Navas, and Roch (2014), and De-Paz, Marín-Solano, and Navas (2013)

The following Corollary is the standard result in the Ramsey model with exponential discount and will prove useful for Proposition 1.

Corollary 1. *For any σ, the weighted average of instantaneous rates of time preferences under the exponential discount function, λ^{ces}_{Exp}, collapses to $\hat{\rho}$, i.e., $\lambda^{ces}_{Exp} = \hat{\rho}$.*

Proof. See Online Appendix B.5. □

Thus, it is easy to see that when agents have either *logarithmic utilities* or the *canonical exponential discount function*, all the time-inconsistency behavior disappears, and agents will stick to the plan. By contrast, Proposition 1 shows that in equilibrium (when the interest is constant) once the elasticity of intertemporal substitution differs from one ($\sigma \neq 1$), and agents have non-constant discounting, the effective rate $\lambda^{\text{ces}}_{\theta}$ can be higher or lower than $\lambda^{log}_{\theta} = \hat{\rho}$, restoring the behavioral differences that motivate our analysis.

Proposition 1. *Under CES utility and a non-constant discount function, for $\sigma < 1$ we have $\lambda^{ces}_{\theta} > \hat{\rho}$. Conversely, $\lambda^{ces}_{\theta} < \hat{\rho}$, if $\sigma \in (1, \bar{\sigma})$, where $\bar{\sigma} = \bar{\sigma} = \frac{r}{r-\rho_\infty} > 1$.*

Proof. See Appendix A.1. □

This observation is critical for the rest of the paper. Once preferences deviate from the logarithmic benchmark, the effective rate of time preference $\lambda^{\text{ces}}_{\theta}$ no longer coincides with $\hat{\rho}$; its position relative to $\hat{\rho}$ depends on whether the elasticity of intertemporal substitution (EIS) or the inverse of the constant relative risk aversion (CRRA) is above or below one.[26] Specifically, when $\sigma > 1$ households behave *as if* they were *more patient* than under exponential discounting, saving more which leads to higher growth rates in the future, whereas for $\sigma < 1$ they behave *as if* they were *less patient* or more "impulsive".[27]

The economic intuition behind this result captures the conflict between the household's long-term saving strategy, shaped by the EIS (σ), and the short-term temptation of present bias. The parameter σ captures the household's willingness to substitute consumption intertemporally; its inverse, $1/\sigma$, reflects the desire for consumption smoothing.

When $\sigma < 1$, the elasticity of substitution is low, implying a strong preference for consumption smoothing. The optimal long-term plan for such a household involves saving just enough to have a relatively flat consumption path. The marginal reward for additional saving is therefore modest. When

[26] A direct comparison with the purely sophisticated agent —who is time-consistent— would imply a similar Proposition, showing the ordering of the effective rate of time preference for the sophisticated agent versus the one for the naive agent, but all the rest of the paper would remain unchanged. We believe this is a parsimonious way to show the main mechanism of present bias in a general equilibrium model, where the *identical overall impatience* is used and compare *constant* vs *non-constant discounting* (Strulik, 2015; Cabo et al., 2015, 2020a,b).

[27] I thank Sharat Ganapi for the interpretation of this "impulsive" vs "saving too much" behavior for σ below or above one.

faced with the immediate gratification offered by present bias, this modest long-term gain is easily outweighed. Households give in to the temptation, save less than a time-consistent agent would, and thus act *as if* they are more impatient. This behavior is captured by a higher effective rate of time preference ($\lambda_\theta > \hat{\rho}$) as shown above.

Conversely, when $\sigma > 1$, the elasticity of substitution is high. This implies a low preference for smoothing and a high willingness to adjust consumption over time to capitalize on returns. The optimal plan involves aggressive saving today to finance a steeply rising consumption path in the future. The long-term reward for this patience —a higher level of future consumption— is a powerful motivator that provides a strong incentive to resist the short-term temptation of present bias. The ambition to achieve high future growth can dominate the immediate impulse to consume, leading households to save even more than their time-consistent counterparts. They behave *as if* they are more patient, which is reflected in a lower effective rate of time preference ($\lambda_\theta < \hat{\rho}$).

These contrasting behavioral responses are the driving force behind the general equilibrium outcomes of the model. In the next sections, we will show that when $\sigma > 1$, time-inconsistent agents will foster faster capital accumulation, extract less resource, and experience higher rate of idea creation, while when $\sigma < 1$ may produce the opposite results for the labor force allocated in the R&D sector, the growth rates of new blueprint, the extraction of the resource and the growth rate of pollution. However, under $\sigma < 1$, we will see that the growth rate of the economy under time-inconsistent agents is always lower. These outcomes are developed in Theorem 1 and explored quantitatively in Section 5, showing the mechanism through which time-inconsistent preferences reshape growth, innovation, and environmental outcomes.

4 General (Market) Equilibrium

An allocation in this economy is defined by the time paths of consumption, the total number of varieties, the extraction of the natural resource, $[C(s), N(s), R(s)]_{s=t}^{\infty}$; time paths of quantities of each intermediate good and prices, $[x(\nu, s), p^x(\nu, s)]_{s=t}^{\infty}$; the time paths of wages, interest rate, and price of the natural resource, $[w(s), r(s), p_R(s)]_{s=t}^{\infty}$, and its corresponding pollution $[P(s)]_{s=t}^{\infty}$.

Furthermore, the equilibrium $\mathscr{E}$ in this economy is defined by an allocation previously defined where all monopolists choose their quantities and price of the intermediate machine $[x(\nu, s), p^x(\nu, s)]_{s=t}^{\infty}$ to maximize their discounted profits; the evolution of prices clean the market $[w(s), r(s), p_R(S)]_{s=t}^{\infty}$; the evolution of the number of varieties $[N(s)]_{s=t}^{\infty}$ is determined by the free entry condition; and the evolution of $[C(s), R(s)]_{s=t}^{\infty}$ is consistent with the household and resource problem.[28]

[28] Due to the market power of the monopolist, and pollution externalities, the equilibrium is not competitive. However, the primary objective of this paper is not to study optimal allocations. Instead, the central research question delves into exploring

Substituting the demand for machinery (9) and machine prices (12) into the production function (4) yields

$$Y(s) = \frac{1}{1-\beta_1-\beta_2}\left(\frac{r}{1-\beta_1-\beta_2}\right)^{-\frac{1-\beta_1-\beta_2}{\beta_1+\beta_2}} N(s)L_F(s)^{\frac{\beta_1}{\beta_1+\beta_2}}\ R(s)^{\frac{\beta_2}{\beta_1+\beta_2}}. \tag{26}$$

Following Scholz and Ziemes (1999), we assume that monopolists have the following capital production $x(\nu,s) = k(\nu,s)$ for all $\nu \in [0, N(s)]$, which describes a linear relationship between the output of the monopolist ν, $x(\nu,s)$, and the input man-made capital, $k(\nu,s)$. Thus, using the production function of the monopolist $x(\nu,s) = k(\nu,s)$, the final production function can be rewritten in equilibrium as,

$$Y(s) = \frac{1}{1-\beta_1-\beta_2}K(s)^{1-\beta_1-\beta_2}\left[N(s)L_F(s)\right]^{\beta_1}\ \left[N(s)R(s)\right]^{\beta_2}, \tag{27}$$

where $K(s) \equiv N(s)\ x(\nu,s) = N(s)\ k(\nu,s)$ and $k(\nu_1,s) = k(\nu_2,s) = k(s)$. From this expression, it can be seen that when $N(s)$ increases, the productivity of labor and the productivity of the resource also increase.

4.1 Properties of the Balanced Growth Path

We now study the Balanced Growth Path (BGP) of the economy. Using the Hotelling rule (3) from the natural resource sector, the FOC of the monopolist ν eq. (B.6), together with log-differentiating (8) and obtaining the growth rate for the prices of the resource, one gets

$$\underbrace{r(s) = \gamma_{p_R}}_{\text{Hotelling eq.(3)}} = \gamma_Y - \gamma_R.$$

Making use of eq. (B.6), and the definition of $K(s)$, the following condition holds,

$$\underbrace{\gamma_Y - \gamma_R}_{r} = \underbrace{(1-\beta_1-\beta_2)}_{\text{Inefficiency terms}} K(s)^{-\beta_1-\beta_2}\left[N(s)L_F(s)\right]^{\beta_1}\left[N(s)R(s)\right]^{\beta_2}. \tag{28}$$

From the final production function (27), one gets the expression for the economy's marginal productivity of total capital,

$$\frac{\partial Y(s)}{\partial K(s)} = K(s)^{-\beta_1-\beta_2}\left[N(s)L_F(s)\right]^{\beta_1}\ \left[N(s)R(s)\right]^{\beta_2}. \tag{29}$$

the impact of time-inconsistency, a facet of human behavior, within the economy. Hence, the focus of this paper does not lie in investigating Pareto optimality or rectifying various externalities. Rather, its aim is to examine how time-inconsistency influences the *laissez-faire* equilibrium.

As pointed out by the Scholz and Ziemes (1999, p.176), the market solution violates the Solow-Stiglitz condition, i.e., $\gamma_Y - \gamma_R = K(s)^{-\beta_1-\beta_2}\left[N(s)L_F(s)\right]^{\beta_1}\left[N(s)R(s)\right]^{\beta_2}$, where the RHS would correspond to the interest rate, and marginal productivity of capital in the social planner problem. One can notice that there is a mismatch between the efficient solution and condition (28), diminished by the term $(1-\beta_1-\beta_2)$. As discussed by the mentioned authors, "[t]his inefficiency is caused by the monopolistic sector that pays an interest rate lower than the marginal productivity of capital". Thus,

$$\underbrace{\gamma_{p_R} = r = \gamma_Y - \gamma_R}_{\text{Marginal Productivity of the Resource}} < \underbrace{K(s)^{-\beta_1-\beta_2}\left[N(s)L_F(s)\right]^{\beta_1}\left[N(s)R(s)\right]^{\beta_2}}_{\text{Marginal Productivity of Capital}} \tag{30}$$

The economic interpretation of eq. (30) is as follows. The left-hand side represents the additional production obtained by not extracting one additional unit of the resource today and extracting it in the future. The right-hand side shows the additional production that could be reached in the future by obtaining an additional unit of the resource today. This extra extraction would be used in production, and adding this extra unit to the capital stock will increase the output in the future.

The next proposition shows the equilibrium conditions of the BGP.[29]

Proposition 2. *The equilibrium conditions along the BGP are given by*

$$\gamma_{\frac{c}{K}} = \frac{Y(s)}{K(s)}\left[\sigma(1-\beta_1-\beta_2)^2 - 1\right] - \sigma\lambda_\theta + \frac{c(s)}{K(s)}, \tag{31}$$

$$\gamma_R = (\beta_1+\beta_2)\frac{Y(s)}{K(s)} - \frac{c(s)}{K(s)} + \eta(1-L_F(s))\frac{\beta_2}{1-\beta_1-\beta_2} + \eta L_F(s)(\beta_1+\beta_2), \tag{32}$$

$$\begin{aligned}\gamma_{L_F} = (\beta_1+\beta_2)\frac{Y(s)}{K(s)} - \frac{c(s)}{K(s)} + \eta(1-L_F(s))\left[\frac{\beta_2-(1-\beta_1-\beta_2)}{1-\beta_1-\beta_2}\right] \\ +\eta L_F(s)\frac{(\beta_1+\beta_2)(1-\beta_2)}{\beta_1},\end{aligned} \tag{33}$$

$$\gamma_{\frac{Y}{K}} = (\beta_1+\beta_2)\eta L_F(s) - (1-\beta_1-\beta_2)(\beta_1+\beta_2)\frac{Y(s)}{K(s)} + \frac{\beta_2}{1-\beta_1-\beta_2}\eta\left(1-L_F(s)\right), \tag{34}$$

and the dynamics of pollution is given by

$$\gamma_P(s) = \mu_2\underbrace{\eta(1-L_F(s))}_{=\gamma_N(s)} + \mu_2\frac{\beta_1}{\beta_1+\beta_2}\cdot\gamma_{L_F}(s) + \left(\mu_1+\mu_2\frac{\beta_2}{\beta_1+\beta_2}\right)\left(\underbrace{\gamma_{\frac{R}{S}}(s) - \frac{R(s)}{S(s)}}_{=\gamma_R(s)}\right), \tag{35}$$

where $\gamma_R(s)$ is defined in eq. (32).

[29] It is straightforward to get the BGP conditions for the log utility case. One should write subscripts θ in the previous proposition to note that all the variables depend on the discount function. However, we avoid it to simplify notation.

Proof. See Appendix A.2. □

At every moment, the BGP for this economy is described by the system of differential equations in Proposition 2. Jointly solved, these equations give information on the steady state and properties of its stability.

4.2 Properties of the Steady State

In the steady state, consumption, net final output, and aggregate capital grow at the same rate γ.[30] Moreover, labor used in the final production sector remains constant, while pollution can follow one of three scenarios: it may keep rising, stay constant (either at a positive or zero level), or decline. We therefore distinguish three theoretical pollution regimes: i) *UN / Humankind Goal* ($\gamma_P < 0$), where pollution falls over time; ii) *Business-as-Usual (BAU)* ($\gamma_P = 0$), where pollution remains flat; and iii) *Crazy Humans will Kill Themselves (CHKT)* ($\gamma_P > 0$), where pollution continues to rise.

Proposition 3. *In the steady state, the labor force used in the final good sector $L_F^{*,\theta}$, the growth rate of the extraction of the exhaustible resource γ_R^θ, the growth rate of the new different number varieties of inputs γ_N^θ, the growth rate of the entire economy γ^θ, and the growth rate of pollution are given by*

$$\gamma^\theta = H_2 \times \{\eta \cdot (\beta_1 + \beta_2) A_0 - \lambda_\theta \cdot [\beta_1 + \beta_2 + \beta_2(1 + A_0)]\}, \tag{36}$$

$$L_F^{\theta,*} = \frac{\eta + \lambda_\theta - \left(\frac{\sigma-1}{\sigma}\right)\gamma^\theta}{(1 + A_0)\eta}, \tag{37}$$

$$\gamma_R^\theta = \eta - (1 + A_0)\eta \cdot L_F^{\theta,*}, \tag{38}$$

$$\gamma_N^\theta = \eta \frac{A_0}{1 + A_0} - \frac{\lambda_\theta - \left(\frac{\sigma-1}{\sigma}\right)\gamma^\theta}{1 + A_0}, \tag{39}$$

$$\gamma_P^\theta = \mu_2 \cdot \underbrace{\eta(1 - L_F^{\theta,*})}_{\equiv \gamma_N^\theta} + \left(\mu_1 + \mu_2 \frac{\beta_2}{\beta_1 + \beta_2}\right)\gamma_R^\theta, \tag{40}$$

where $A_0 = \frac{(1-\beta_1-\beta_2)(\beta_1+\beta_2)}{\beta_1} \in (0, \infty)$, due to the constraints $\{0 < \beta_1 < 1\} \wedge \{0 < \beta_2 < 1\} \wedge \{0 < \beta_1 + \beta_2 < 1\}$ and $H_2 = \frac{\sigma}{\beta_1 + A_0(\sigma\beta_1 + \beta_2) + \beta_2(2-\sigma)}$ is positive.

[30] This was noted in Kaldor (1961), where he observed that output per capita increases, while the ratio capital/output remains practically constant.

Proof. See Appendix A.3. When utility is *logarithmic* ($\sigma = 1$), the steady-state relationships simplify considerably: the labor share allocated to final-good production, the growth rate of resource extraction, and the growth rate of intermediate varieties are all *independent* of the overall growth rate of the economy. This contrasts with the general CES specification, in which these variables remain endogenously linked to the aggregate growth. □

Plugging in the corresponding growth rates γ^θ into $L_F^{\theta,*}$, γ_N^θ and γ_R^θ gives their corresponding steady state values, which are constant over time. Also note that all of growth rates above in the steady state, are functions of the deep parameters of the model, that is, $\gamma^\theta = \gamma^\theta(\beta_1, \beta_2, \eta, \sigma, \lambda_\theta)$, $L_F^{\theta,*} = L_F^{\theta,*}(\beta_1, \beta_2, \eta, \sigma, \lambda_\theta)$, $\gamma_R^\theta = \gamma_R^\theta(\beta_1, \beta_2, \eta, \sigma, \lambda_\theta)$, $\gamma_N^\theta = \gamma_N^\theta(\beta_1, \beta_2, \eta, \sigma, \lambda_\theta)$, the growth rate of pollution has two extra parameters governing its dynamics, $\gamma_P^\theta = \gamma_P^\theta(\mu_1, \mu_2, \beta_1, \beta_2, \eta, \sigma, \lambda_\theta)$. The condition that must hold in order to get a positive growth shows that the weighted mean of the instantaneous discount rates λ_θ should be lower than a given threshold,

$$\lambda_\theta < \eta \cdot \frac{(\beta_1 + \beta_2)\, A_0}{[\beta_1 + \beta_2 + \beta_2(1 + A_0)]}.$$

Since the interest rate is constant in the steady state, one can take logs in expression (13) and differentiate with respect to time, and noting that the growth rate of the labor force used in the final good sector in the steady state is zero, it leads to

$$(\beta_1 + \beta_2) \cdot \gamma_{x(\nu,s)}^\theta = \beta_2 \cdot \gamma_R^\theta. \tag{41}$$

As the growth rate of the extraction is negative, it implies that $\gamma_{x(\nu,s)}^\theta$ is also negative, which means that the growth rate of the demand for a given intermediate good $\nu \in [0, N]$ will decline over time.

Remark 2. *The decline in the extraction rate of the resource (expressed by γ_R^θ) is proportional to the decline in the demand for intermediate good $\nu \in [0, N]$ (expressed by γ_x^θ),*

$$\left(\frac{\beta_2}{\beta_1 + \beta_2}\right) \gamma_R^\theta = \gamma_x^\theta.$$

Foreseeing a persistent decline in demand, each intermediate-goods monopolist ν will decrease their production to keep prices $p^x(\nu, s) = p^x \in \mathbb{R}_+$ constant over time, as shown in eq. (12). Plugging (41) into the growth rate of the final production given in (A.27), we get the growth rate of the economy γ^θ as the sum of the growth rate of the new different number varieties of inputs γ_N^θ, which is positive, and the growth rate of the production of the monopolist ν, $\gamma_{x(\nu,s)}^\theta$, which is negative:

$$\gamma^\theta = \underbrace{\gamma_N^\theta}_{>0} + \underbrace{\gamma_{x(\nu,s)}^\theta}_{<0}. \tag{42}$$

If rate of new entrants (new monopolists) in the market (γ_N^θ) is greater than the rate of decrease in the supply of a individual monopolist ($\gamma_{x(\nu,s)}^\theta$), the economy will have a positive growth rate. Consequently, there will be growth as long as new monopolists enter the market to compensate for the decrease in supply by the old monopolists. If this will not be the case, then the growth rate will be negative and the economic activity will disappear in the long run.

We now focus on the evolution of the natural resource $S^\theta(s)$, its extraction $R^\theta(s)$, and the extraction rate $\frac{R^\theta(s)}{S^\theta(s)}$.

Lemma 1. *The extraction of the resource $R^\theta(s)$, evolves over time as, $R^\theta(s) = R_0 \exp\left\{\gamma_R^\theta \cdot s\right\}$, where $\gamma_R^\theta < 0$, and the initial extraction rate $R_0^\theta \in \mathbb{R}_+$ is determined by $R_0^\theta = -\gamma_R^\theta \times S_0$. The amount of exhaustible resource left $S^\theta(s)$, and the extraction rate $R^\theta(s)/S^\theta(s)$ are given by*

$$S^\theta(s) = S_0 \exp\left\{\gamma_R^\theta \cdot s\right\}, \qquad \frac{R^\theta(s)}{S^\theta(s)} = -\underbrace{\gamma_R^\theta}_{<0} > 0. \tag{43}$$

Proof. See Appendix A.4 □

This shows that the extraction rate $R(s)/S(s)$ will be constant in this economy.

Corollary 2. *The extraction of the resource can be represented as a Markovian strategy, where the control is a function of the state variable*[31]

$$\phi^\theta(S) \equiv R^\theta(S) = -\gamma_R^\theta \times S^\theta. \tag{44}$$

For a graphical intuition, see the simulations in the quantitative analysis section. In the logarithmic utility case, the negative instantaneous discount rate of exponential discounting, $(-\hat{\rho})$, coincides with both the steady-state growth rate of resource extraction, $\gamma_R^{\theta,\log}$, and the negative "effective rate of time preference" $\lambda_\theta^{\log}$ (see Remark 1 for the second result). This one-to-one correspondence breaks down under CES utility, where $\hat{\rho}$, $\gamma_R^{\theta,\text{ces}}$, and $\lambda_\theta^{\text{ces}}$ generally diverge.

Corollary 3. *In steady state, and for any discount function $\theta(j)$, the growth rate of resource extraction satisfies*

$$\gamma_R^{\theta,h} = \gamma^{\theta,h} \cdot \left(\frac{\sigma - 1}{\sigma}\right) - \lambda_\theta^h \quad \text{for } h \in \{log, ces\}. \tag{45}$$

[31] However, as highlighted by Groth (2015), "along any economic development path, the aggregate input of non-renewable resources must in the long run asymptotically approach zero. From a physical point of view, however, there must be some minimum amount of the resource below which it can not fulfill its role as a productive input. Thus, strictly speaking, sustainability requires that in the "very long run", non-renewable resources become inessential".

Proof. See equation (A.28) in the Appendix. □

Corollary 4. *Under logarithmic utility, the Markovian extraction collapses to the same strategy regardless of the discount function $\theta(j)$, i.e., $\phi^{log}(S) \equiv R^{log}(S) = -\gamma_R^{log} \cdot S^{log} = \hat{\rho} \cdot S^{log}$.*

Proof. All that needs to be proved is that $\gamma_R^{log} = -\hat{\rho}$. See Online Appendix B.7. □

This result uncovers an interesting equivalence. With logarithmic utility, the steady-state growth rate of resource extraction collapses to $\gamma_R^{log} = -\hat{\rho}$, the constant rate of time preference that defines exponential (time-consistent) discounting. Through general-equilibrium feedback, the growth rate of the resource depletion chosen by the resource firm therefore mirrors precisely the way households would value the future under a fixed, exponential discount rate.

The next corollary formalizes the link between households' intertemporal elasticity of substitution and the resource extraction rate: the larger the elasticity (i.e., the greater households' willingness to shift consumption over time), the higher the firm's extraction rate. A visual counterpart of this result is provided in Figure C.6.

Corollary 5. *Monotonicity in σ. The lower the CES, the higher the extraction rate of the resource.*

$$\frac{R^{\theta}_{\sigma>1}(s)}{S^{\theta}_{\sigma>1}(s)} < \frac{R^{log}(s)}{S^{log}(s)} < \frac{R^{\theta}_{\sigma<1}(s)}{S^{\theta}_{\sigma<1}(s)} \tag{46}$$

Proof. See Online Appendix B.8. □

However, if households discount the future exponentially (hence time-consistent behavior), the economy has a higher extraction rate if the EIS is lower than one. On the other hand, if the EIS is greater than one, the economy populated by time-consistent agents with exponential discounting has a higher extraction rate.

Corollary 6. *An economy populated by exponential discounters has a lower extraction rate than an economy populated by time-inconsistent agents if the EIS is less than one. On the other hand, if the EIS is greater than one, the economy with time-consistent agents discounting exponentially has a higher extraction rate of the resource. That is,*

$$\frac{R^{\sigma<1}_{Exp}(s)}{S^{\sigma<1}_{Exp}(s)} < \frac{R^{\sigma<1}_{\theta}(s)}{S^{\sigma<1}_{\theta}(s)}$$

with the inequality reversed if $\sigma > 1$.

Proof. See Online Appendix B.9. □

In Section 5 we provide numerical and graphical illustrations of the previous results. Section B.11 studies the stability properties of the steady state.

Importantly, one can show what is the general effect of moving from an economy with consistent preferences to one with time-inconsistent preferences in the following result.

Theorem 1 (Impact of Time-inconsistency on Steady-State Economic and Environmental Outcomes). *Moving from an economy populated by time-consistent (TC) agents to one with time-inconsistent (TI) households has a profound effect on all the important steady-state growth rates and the allocation of scientists in the economy.*

If $\sigma > 1$, TI agents enjoy higher growth rates; devote fewer workers to final goods production (and hence more scientists to R&D); create new ideas faster; extract the resource less aggressively (meaning a slower rate of resource depletion); and the growth rate of pollution is higher.

If $\sigma < 1$, TI agents experience lower economic growth. However, the effect on labor allocation (and thus R&D effort), innovation, resource extraction, and pollution growth is ambiguous. It may increase, decrease, or remain unchanged relative to the TC benchmark.

In the special case of logarithmic utility ($\sigma = 1$), time-inconsistency has no impact on the economy.

Proof. See Appendix A.5. See Table A.1 for a concise summary of the results. □

Thus, when agents in an economy exhibit logarithmic utility ($\sigma = 1$), temporal preferences become irrelevant and one cannot capture the empirical evidence shown by the behavioral scientists cited above. Consequently, there are no changes in the key growth rates or other variables of interest: the steady-state allocation of labor, innovation rate, resource extraction path, and pollution growth all coincide with the time-consistent benchmark. Thus, an economy with present bias but logarithmic utility behaves identically —i.e., is observationally equivalent— to an economy where agents do not exhibit present bias.

The unique way to obtain meaningful macroeconomic effects is if $\sigma \neq 1$, and the sign of $(\sigma - 1)$ then dictates whether TI amplifies or slows down growth, research effort, the creation of ideas, resource use, and pollution. In particular, for $\sigma > 1$ the TI economy grows faster, shifts labor toward R&D, innovates more rapidly, depletes the resource more slowly, and generates pollution at a higher rate.

However, if $\sigma < 1$, TI agents experience lower growth rates; may allocate more, fewer, or the same number of workers in the final sector (and thus allocate less, more or the same number of scientist to the R&D sector); may create new ideas slower, faster or at the same speed; may extract the resource more or less aggressively, or at the same rate (leading to faster, slower, or unchanged depletion); and the growth rate of pollution can be higher, lower, or unchanged compared to an economy populated by TC agents.

Hence it is not generally valid to claim that time-inconsistent agents will always allocate more scientists to research and development, extract the resource more gently, or have lower pollution growth

even when the EIS is less than one ($\sigma < 1$). It will depend on the specific parameters of the model.

Consequently, the lower economic growth observed under time-inconsistent preferences with an EIS lower than one is driven by the fact that it is affected directly by the effective rate of time preference, λ_θ, which is higher when EIS < 1, capturing the idea that agents behave *as if they are more impatient*. This higher effective rate of time preference reduces growth directly. Consequently, under this case, an economy with present-biased households would fail to converge. On the contrary, with an EIS greater than one, an economy with present-biased households would grow faster and converge to the richest economies.

Welfare analysis in this setting is inherently delicate, because exponential —with no present bias— and non-exponential discounters —with present bias— rank intertemporal allocations according to different discounted utility functionals. Simply comparing the discounted sums of utilities along their respective optimal paths would therefore mix incompatible welfare metrics, unless one first imposes an explicit normative social welfare criterion (for instance, a common exponential discount rate $\hat{\rho}$ chosen via the observational equivalence condition in (21)). Rather than taking a stand on which intertemporal welfare index should be considered normatively correct, we focus on the model's normative outcomes —economic growth, resource extraction, innovation, and pollution— highlighting how time-inconsistency reshapes these key variables.

5 Quantitative Exploration

We now calibrate the model to quantitatively explore the theoretical predictions derived in Section 4, particularly those summarized in Theorem 1. Our primary goal is to show how the interaction between time-inconsistent preferences shapes long-run economic and environmental outcomes.

Following the environmental macro literature, the nonrenewable resource is used as an *energy input* rather than for instance, using oil or gas per se to produce a final good (Golosov et al., 2014; Hassler and Krusell, 2012; Jones and Vollrath, 2024). As in Golosov et al. (2014) and Hassler and Krusell (2012), we set the resource elasticity parameter $\beta_2 = 0.04$.[32] We set the elasticities of the intermediate machines to $1 - \beta_1 - \beta_2 = 0.4$ (see e.g., Englander and Gurney (1994) and Maddison (1987)). The previous implies an elasticity of the labor force used in the final sector of $\beta_1 = 0.66$.

Jones (2016a) elegantly captures the debate over the EIS, an important parameter in our model. He writes that "[i]n the most common way of specifying preferences for macro applications, the coefficient of relative risk aversion, $[1/\sigma]$ in our notation, equals the inverse of the elasticity of intertemporal

[32] See the World Bank data on "Total natural resources rents (% of GDP)" https://data.worldbank.org/indicator/NY.GDP.TOTL.RT.ZS. The world total natural resources rent for 2021 was a bit higher at a 3%. However, it varies widely, ranging from 0.2% in the European Union, to 30% in Angola, 43.3% in Iraq, and 61% in Libya.

substitution. Large literatures on asset pricing (Lucas 1994) and labor supply (Chetty 2006) suggest that $[1/\sigma > 1]$ is a reasonable value, and values above 1.5 are quite common in this literature. Evidence on the elasticity of intertemporal substitution, $[\sigma]$ in our notation, is more mixed. The traditional view, such as Hall (1988), is that this elasticity is well below one, consistent with the case of $[1/\sigma] > 1.3$. This view is supported by a range of careful microeconometric work, including Attanasio and Weber (1995), Barsky et al. (1997), and Guvenen (2006); see Hall (2009) for a survey of this evidence. On the other hand, Vissing-Jorgensen and Attanasio (2003) and Gruber (2006) find evidence that the elasticity of intertemporal substitution is greater than one, suggesting that $[1/\sigma] < 1$ could be appropriate." Thus, he shows that there is "mixed evidence" regarding the particular estimation of such a deep parameter, although in the macro literature, studies always assume that the EIS is lower than one. Additionally, in a recent paper by Charles Jones focusing on the effects of AI in the macroeconomy, he also studies the cases of the CRRA being higher or lower than one (Jones, 2024). Flynn et al. (2023) likewise stress that whether the EIS is above or below unity can fundamentally alter the qualitative implications of many models. [33]

Another paper looking at the EIS below and above one is Cai and Lontzek (2019), where the authors study a dynamic stochastic integration of climate and economy, and "examine a wide but plausible range of values for critical parameters". However, they rely on the Epstein-Zin utility specifications of dynamic preferences, which can clearly disentangle risk aversion and the intertemporal elasticity of substitution, claiming that "[c]urrent empirical analyses do not give us precise estimates of critical parameters, particularly risk aversion and the intertemporal elasticity of substitution". They simulate their macroeconomic model for values of the EIS ranging from 0.5 to 2.0 and the risk aversion from 2 to ∞ (see their Tables 3 to 10). Recently, Maxted (2025), also studies the effect of an EIS above or below one. Additionally, Krusell et al. (2002) also consider values of the EIS between $1/7$ and $1/0.5 = 2$.

As noted in Cabo et al. (2020b), it is normal to assume in theoretical models of economic growth a

[33] As Flynn et al. (2023) put it, "This lack of consensus regarding the magnitude of EIS is troubling since the relationship between EIS and unity plays a central role in affecting the dynamics of many theoretical models in both quantitative and *qualitative* terms. For example, in the Bansal and Yaron (2004) long-run risk model, when EIS > 1, investors are willing to pay a premium to hedge against lower future economic growth rates. The wealth-consumption ratio is pro-cyclical, the equity premium is high, and the risk-free rate is low and stable. Setting EIS < 1 changes many basic intuitions for the model and often reverses each of these properties.(Similar changes occur to asset prices and quantity dynamics in production-based models (Kaltenbrunner and Lochstoer 2010; Croce 2014)). Moreover, Kaplan and Violante (2014) find that EIS > 1 is crucial for getting households to hold large illiquid positions and thus to quantitatively match the consumption response to tax rebates. Assuming EIS > 1 also has striking implications for the response of asset prices to changes in uncertainty. (In endowment economies, Bansal and Yaron (2004) and Barro (2009) find that asset prices fall in response to increased volatility and disaster risk when EIS > 1, and rise otherwise. Drechsler and Yaron (2011) and Di Tella (2017) argue that EIS > 1 is crucial for explaining the variance risk premium and balance-sheet recessions, respectively.)"

log-utility, as this simplifies the analysis when one studies "complexities like non-constant time preferences". However, whether the EIS value σ should be greater or lower than one has been called into question as shown above. As pointed out in Thimme (2017), the default assumption in macroeconomic models of using an EIS between 0.5 and 1 (prevailed by Hall (1988)'s early EIS estimates close to zero) has been challenged by recent literature. For instance, Havranek et al. (2015) and Gruber (2013) mention values of the constant elasticity of intertemporal substitution σ greater than one. Values greater than one are correlated with higher wealth, education and are country specific (see Havranek et al. (2015), Ben-Gad (2012), and Thimme (2017)). Acharya et al. (2023) and Gollier et al. (1997) also show that the EIS $\sigma = 1$ constitutes a threshold, at which the behavior of naive agents reverses, thus, giving the opposite results if it is above or below one. As we have shown in Theorem 1, the naive agent will enjoy a higher final output —and thus consume more—, have more ideas, and extract the resource less aggressively if and only if her EIS or the inverse of the coefficient of relative risk-aversion is above one,and an economy populated by time-inconsistent agents with an EIS below one ($\sigma < 1$) will consume less for sure, but it is ambiguous if they will create fewer ideas, or extract more resource (see also Table A.1 for a concise summary of the results). Motivated by this, we study $\sigma \in \{\frac{1}{1.5}, 1, 1.2\}$ as representative "below-one, log, above-one" cases. The log case serves as a neutral benchmark, where time-inconsistency has no effect on the economy (see Theorem 1).

Regarding the natural resource aspect of the model, we follow Hassler and Krusell (2018), where they write: "the total estimated reserves of conventional oil measured in carbon content is around 300 GtC [...]. For conventional natural gas a reserve of 200 GtC is estimated. The total use of fossil fuel is about 10 Gt per year". This implies a resource extraction of $10/(300+200) = 0.02$. Therefore, as under the baseline case ($\sigma = 1$), the equilibrium extraction rate is $rs^* = -\gamma_R^* = \hat{\rho}$, we use the discount rate for the exponential discount function to be $\hat{\rho} = 0.02$, to capture this stylized fact. However, one should note that under an EIS different than one, $rs^* = -\gamma_R^*$ does not coincide with the constant discount rate (see Lemma 4). We normalize to teratonnes of carbon (TtC), where 1 TtC $= 10^{12}$ tC $=$ 1000 GtC. Thus, $S_0 = 0.5$ TtC. The fact that $\hat{\rho} = 0.02$ is in line with the "Medium Future" (26 to 75 years) value in the data by Weitzman (2001).

With the share of narrowly defined labor force allocated to R&D activities, empirical values range from 0.5% to 1.5% (see e.g., Jones (1995, 2016b)). For instance, Jones (2016b) reports that "[i]n particular, these data only capture a small part of what an economist would call research. For example, around 70% of measured R&D occurs in the manufacturing industry. In 2012, only 18 million workers (out of US employment that exceeds 130 million) were employed by firms that conducted any official R&D." Using this broader proxy implies $L_{RD}^* = 18/130 = 14\%$, giving $L_F^* = 86\%$, better reflecting the wider ecosystem that contributes to creation of ideas. Thus, following Charles Jones broader definition, we

would allow L_F^* to be between 80% and 95%.

To calibrate η, we match the growth rate of the economy under the benchmark/neutral scenario with $\sigma = 1$, which is 1.8% per year –see e.g., Hassler and Krusell (2018)–, and with log utilities the effects of time-inconsistency disappears (see Remark 1). Note that η also determines the labor force in the R&D sector, showing that the growth rates of new intermediate inputs can be written as $\gamma_N^\theta = \eta(1 - L_F^\theta(\beta_1, \beta_2, \eta, \sigma, \lambda_\theta))$. This gives a value of $\eta = 0.142163$. With this value, $L_F = 86.53\%$. Sensitivity analysis performed around this parameter shows that choosing a lower η leads to a lower growth rate of the economy, but at the same time a lower Y/K (i.e., higher K/Y) showing the tension between the two, the growth rate of the economy and the capital-output ratio. One could calibrate η to match a higher capital-output ratio (lower output-capital ratio), but then the growth rate of the economy would be lower. Although recognizing the debate on the endogenous growth literature and the vast work by Charles Jones and coauthors on whether ideas are harder to find (see for instance Bloom et al. (2020)), here we decided to follow the basic Romer model, and focus on the effects of *present bias* on the whole economy.

To calibrate for the initial N_0, we start with the $EIS = 1$ benchmark case, using the initial labor in the final sector $L_{F,0} = 86.53\%$ (see Table 2 below, as η is calibrated to generate a 1.8% growth rate under the $\sigma = 1$ scenario), and with the initial global GDP in 2024 of $Y_0 = 111T\$$,[34] it implies an initial $N_0 = 15.0205 \times 10^5$.

For TFP growth, feasible values are between 0.7% and 3.1% per year, which is in line with the historical data for the US economy (Jones, 2016b), and what our model delivers.

Regarding the discount calibration, our benchmark uses exponential discounting with instantaneous discount rate $\hat{\rho} = 0.02$, consistent with "medium future" values in Weitzman (2001). We also consider two discount functions capturing present bias: (i) the hyperbolic discount function following Tsoukis et al. (2017) and Cabo et al. (2020a), and (ii) a convex combination of two exponentials (Karp, 2007; Mañó-Cabello et al., 2021). Note that under exponential discounting, the instantaneous discount rate is constant and equal to $\hat{\rho}$, while under present bias discounting, the instantaneous discount rate varies over time, and this is the reason why exponential discounters are time-consistent, while present bias discounters are time-inconsistent. Online Appendix Table C.3 summarizes the structural parameter values used in the quantitative analysis.

Following Cabo et al. (2020a) and Strulik (2015), we calibrate the present bias specifications so that the *present value* of a unit consumption stream is identical across discount functions, as explained in equation (21). This ensures that all discount functions represent the same overall degree of impatience,

[34] See the World Bank data https://data.worldbank.org/indicator/NY.GDP.MKTP.CD?end=2024&locations=1W&start=1960&view=chart

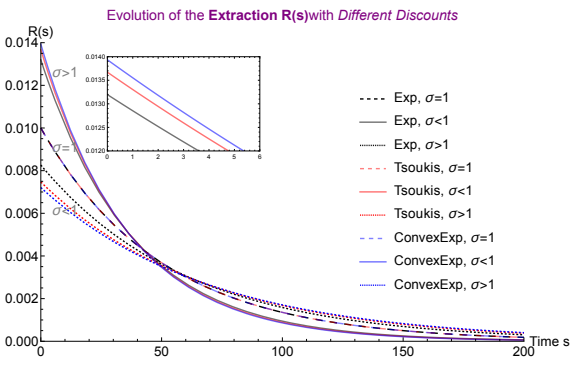
allowing for meaningful comparisons of their effects on the economy. This pins down the admissible parameter sets and ensures like-for-like comparisons with the exponential benchmark. Online Appendix C.4 shows the relationship between the parameters of the Tsoukis et al. (2017) hyperbolic type and convex combination of exponential discount functions and the implied present value, and Table C.4 reports the parameters of the discount functions.

With any discount function $\theta(j)$, the interest rate in equilibrium r is determined endogenously in the model. One should notice that this is not possible to solve for all discounts algebraically. Certainly, one should find the interest rate that satisfies the BGP conditions of $\gamma = \gamma_Y = \gamma_c$. This condition solves r implicitly. We solve it numerically for all the discount functions we consider –and any EIS–, which gives rise to different interest rates in equilibrium. For the logarithmic case one should solve for r using the growth rate of the economy γ and the Euler equations, which gives the growth rate of consumption from the modified Ramsey rule, where λ_θ^{log} collapses to $\hat{\rho}$ for any discount $\theta(j)$. For the CES case, one should solve for r using equations (36) and (23) where λ_θ^{ces} is given by (24). Obviously, all the interest rates for the logarithmic case collapse to the same one. This is straightforward, as for any discount function, with log utilities we have the same consumer behavior (as all the λ_θ^{log} are the same regardless of the discount function, see Remark 1). The outcomes are *observationally equivalent*. All the equilibrium interest rates are shown in Table C.2 in the Online Appendix C.2. It can be seen that the lower the desire to smooth consumption (higher EIS), the lower the interest rate. A further analysis of the interest rates and its effects on the effective rate of time preference $\lambda(\sigma, r)$, and the growth rate $\gamma(\sigma, r)$ —how it affect the Euler equation— is provided in the Online Appendix C.1.

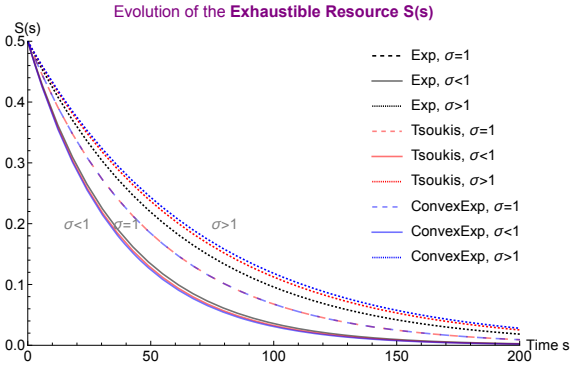
The real return on riskless assets has changed over time, from around 4.5% to 0% in the 2010s (Hassler and Krusell, 2018). Thus, our results are in line with these historical values.

Figure 2 shows the evolution of the extraction strategy of the economy $R(s)$ and the corresponding evolution of the resource $S(s)$ (i.e., see at each point in time how much resource is left when agents behave under all the mentioned discount functions). See Proposition 3 for the growth rate of the extraction strategy under any discount function and any EIS.

When the elasticity of intertemporal substitution is low ($\sigma < 1$), time-inconsistent agents behave *as if they are more impatient*, as established in Proposition 1, and numerically exemplified in Table 1. Through a general equilibrium effect, this higher effective impatience translates into a more aggressive resource extraction strategy in the short run. Consequently, as illustrated in Figure 2, the resource stock for this economy is depleted more rapidly than in the time-consistent logarithmic benchmark, leaving fewer resources available for future generations. This pressure for immediate gratification drives the economy to front-load resource use, pursuing aggressive extraction in the short run at the cost of faster stock depletion. The opposite dynamic unfolds when the EIS is high ($\sigma > 1$). In this case, present-



(a) Evolution of the Extraction Strategy $R(s)$.



(b) Evolution of the Resource Stock $S(s)$.

Figure 2: Evolution of the Extraction $R(s)$ and dynamics of the Natural Resource $S(s)$.

biased agents *act with greater effective patience*, adopting a more conservative extraction path initially and thereby preserving a larger stock of the resource for the long run.

Table 1: Effective Rate of time Preferences, $\lambda_\theta(\sigma)$.

	Exponential	Tsoukis	ConvexExp
CES ($\sigma < 1$)	0.02	0.0213387	0.0220802
Log ($\sigma = 1$)	0.02	0.02	0.02
CES ($\sigma > 1$)	0.02	0.0186176	0.0181622

Note: The table shows the effective rate of time preference values $\lambda_\theta(\sigma)$ for Exponential, Tsoukis, and Convex Exponential discount functions across different EIS (σ) regimes. One can see that the effective rate of time preference is higher for non-constant discounting agents when the EIS is lower than one, and lower when the EIS is higher than one. This is in line with Proposition 1.

The model's steady state yields a constant extraction rate, $R(s)/S(s)$, meaning a fixed percentage of the remaining resource stock is used each period (see eq (43)). The level of this rate, however, depends critically on the interaction between time preferences and the EIS. In the benchmark case of logarithmic utility ($\sigma = 1$), the observational equivalence emerges. As established in Corollary 4, the extraction rate converges to the same value, $R(s)/S(s) = -\gamma_R^{log} = \hat{\rho}$, regardless of whether agents are time-consistent or time-inconsistent. This is shown in the appendix in Figure C.6, where all lines for the log case overlap.

This equivalence breaks down for a general CES utility ($\sigma \neq 1$), but the extraction rate is still the constant $R(s)/S(s) = -\gamma_R^{ces}$ (see Corollary 2). Remember that $\gamma_R^\theta = \gamma^\theta \cdot \left(\frac{\sigma-1}{\sigma}\right) - \lambda_\theta^{ces}$. Here, the extraction rate is determined by the effective rate of time preference, λ_θ. As shown in Table 1, λ_θ varies with the specific discount function, reflecting different degrees of effective impatience. The inverse relationship between the EIS and impatience is clear: for agents with a low desire for consumption

Table 2: Relevant Growth Rates of the Economy

	γ	L_F^*	γ_R	γ_N
Panel A: Low EIS ($\sigma < 1$)				
Exponential	1.28%	89.95%	-2.64%	1.43%
Tsoukis	1.20%	90.45%	-2.73%	1.36%
Convex Exp.	1.16%	90.73%	-2.79%	1.32%
Panel B: Log Utility ($\sigma = 1$)				
All specifications	1.80%	86.53%	-2.00%	1.91%
Panel C: High EIS ($\sigma > 1$)				
Exponential	2.08%	84.68%	-1.65%	2.18%
Tsoukis	2.16%	84.19%	-1.56%	2.25%
Convex Exp.	2.19%	83.98%	-1.52%	2.28%

Note: Growth rates and labor share are expressed in percentages. Panel B collapses all specifications as they yield identical results under log utility.

smoothing ($\sigma > 1$), present bias fosters *effective patience* ($\lambda_\theta < \hat{\rho}$), leading to a more conservative, lower extraction rate. Conversely, for agents with a high desire for consumption smoothing ($\sigma < 1$), present bias manifests as *higher effective impatience* ($\lambda_\theta > \hat{\rho}$). This pressure for immediate gratification drives a more aggressive, higher extraction rate. Generally, it was proved in Corollary 5 that agents extract more (less) with $\sigma < 1$ ($\sigma > 1$) in comparison to the log utility. Table 2 shows the quantitative outcomes. This mechanism directly explains the divergent paths in Figure C.6 and confirms that the behavioral "*Self-Control problem*" (Thaler, 2015, Chapters 11-12) has macroeconomic consequences that are critically mediated by the EIS.

Table 2 summarizes the key steady-state outcomes, providing quantitative validation for our main theoretical predictions in Theorem 1. The results reveal an interesting divergence in economic performance based on the EIS: when the EIS is high ($\sigma > 1$), time-inconsistent (TI) agents achieve "superior outcomes" compared to their time-consistent (TC) counterparts (the agent that discounts exponentially). The economy populated by time-inconsistent agents enjoys higher economic growth (up to 2.19% vs. 2.08%), foster more innovation (higher γ_N), allocates more labor to R&D (lower L_F^*), and conserve the natural resource more effectively (lower depletion rate $-\gamma_R$). On the contrary, if the EIS is low ($\sigma < 1$), the pattern reverses. Time-inconsistency leads to expected results with lower economic growth (as low as 1.16% vs. 1.28%), slower innovation, and a less research-intensive economy, i.e., fewer

workers allocated to the R&D sector.

The resource depletion rate ($-\gamma_R$) offers a particularly clear illustration of the mechanism. In the logarithmic benchmark, the rate is fixed at $-\gamma_R = \hat{\rho} = 2.00\%$. For $\sigma < 1$, that economy extracts more aggressively, but the effect is amplified by present bias: time-inconsistent agents deplete the resource at rates of -2.73% and -2.79%, more agrresively than the -2.64% rate of their time-consistent counterparts. Conversely, for $\sigma > 1$, the economy extracts the resource in a more conservative manner, but again present bias amplifies the tendency: the economy with time-inconsistent agents depletes at a much slower rate of -1.56% and -1.52%, compared to the -1.65% rate for the TC economy. This provides strong numerical support for Corollary 6 (noting that $R/S = -\gamma_R$).

We now turn to the steady-state growth rate of pollution, γ_P. As shown in eq (40), this rate is driven by two opposing forces: the growth of new ideas (γ_N), which increases production and thus pollution, and the rate of resource depletion (γ_R), which is negative and thus eventually constrains it. To explore the full range of possible outcomes, we consider different scenarios of pollution intensity parameters (μ_1, μ_2) to illustrate the three theoretical regimes: declining pollution ($\gamma_P < 0$), stable pollution or "Business-as-Usual" ($\gamma_P = 0$), and rising pollution ($\gamma_P > 0$). The specific calibration of these parameters is detailed in Appendix C.6.

Table C.6 in Online Appendix C.7 presents the steady-state results and reveals the consequence of time-inconsistency for the environment, which is again mediated by the EIS: When the EIS is high ($\sigma > 1$), time-inconsistency leads to a *higher growth rate of pollution* under the rising-pollution scenario, $\gamma_P > 0$ (CHKT). For example, γ_P increases from 1.033% for the time-consistent agent to as high as 1.087% for the time-inconsistent agent. This outcome reflects an interesting trade-off: while time-inconsistent agents deplete resources more slowly (a pollution-mitigating effect) –see Table 2 and Theorem 1–, their significantly higher economic and innovation growth (a pollution-generating effect) dominates, inducing a higher growth rate of pollution under $\sigma > 1$ for an economy populated by time-inconsistent agents. Under the UN scenario, that is, when emissions decline over time ($\gamma_P < 0$), time-inconsistency has the same effect, leading to less pronounced declines in pollution. Here, γ_P moves from -1.383% to a less pronounced decline of -1.260%, reflecting the same trade-off mentioned above.

When the EIS is low ($\sigma < 1$), time-inconsistency results in a *lower growth rate of pollution*. In the rising-pollution scenario, γ_P falls from 0.626% to as low as 0.565%. Here, the trade-off is reversed: the dampening effect of slower economic growth on pollution outweighs the aggravating effect of more aggressive resource extraction.

As always, the logarithmic case ($\sigma = 1$) serves as the neutral benchmark where time preferences have no impact on the long-run pollution path. Appendix C shows the dynamic evolution of pollution

under the declining (UN), constant (BAU), and rising (CHKT) scenarios.

Appendix C.8 shows the time evolution of the main economic variables, allowing us to compare the effect of economies with agents discounting exponentially versus present-biased agents. There we show that When the EIS is high ($\sigma > 1$), the economy populated by time-inconsistent agents consistently outperforms the time-consistent benchmark, achieving higher long-run levels across all key macroeconomic aggregates. In contrast, when the EIS is low ($\sigma < 1$), the conventional intuition holds: the time-consistent economy characterized by rational exponential discounters achieves superior outcomes.

Table 3: Steady-State Ratios and Eigenvalues

	Y/K	c/K	R/S	$\lambda_{Y/K}$	$\lambda_{c/K}$	λ_L
Panel A: Low EIS ($\sigma < 1$)						
Exponential	0.435	0.423	0.026	0.511	0.144	-0.083
Tsoukis	0.437	0.425	0.027	0.515	0.145	-0.083
Convex Exp.	0.438	0.427	0.028	0.516	0.145	-0.084
Panel B: Log Utility ($\sigma = 1$)						
All specifications	0.422	0.404	0.020	0.489	0.142	-0.084
Panel C: High EIS ($\sigma > 1$)						
Exponential	0.415	0.394	0.017	0.477	0.141	-0.084
Tsoukis	0.412	0.390	0.015	0.471	0.140	-0.083
Convex Exp.	0.411	0.388	0.014	0.470	0.139	-0.083

Note: This table shows the steady-state ratios for output-to-capital (Y/K), consumption-to-capital (c/K), and the extraction rate (R/S) which corresponds to the eigenvalue of the extraction over the resource. The last three columns display the eigenvalues (λ) governing the local stability of the system across the different specifications for the EIS (σ) and comapring [constant] vs [non-constant discounting].

Beyond growth rates, the model's steady state is characterized by a set of key economic ratios and specific stability properties. We analyze these to provide a better understanding of the equilibrium structure and its dynamical properties.

As predicted by Lemma 4, the long-run equilibrium is defined by constant ratios of output-to-capital (Y/K), consumption-to-capital (c/K), labor allocated to the final sector (L_F^*), and the extraction rate (R/S). Table 3 reports the values of these ratios, showing how time-inconsistent preferences systematically shift the structure of the economy: when the EIS is low ($\sigma < 1$), the heightened effective impatience of the economy with time-inconsistent agents leads to a steady state with higher ratios of

Y/K and c/K, a larger share of labor allocated to final goods production (higher L_F^*), which implies less labor allocated to R&D, and a more aggressive resource extraction rate (higher R/S). When the EIS is high ($\sigma > 1$), the effective patience of time-inconsistent agents fosters a steady state with lower Y/K and c/K ratios, a greater allocation of labor to R&D (lower L_F^*), and a more conservative resource extraction rate. One could argue that this encapsulates a long-term focus that prioritizes innovation, and resource preservation (as shown above with the growth rates). As always, for the logarithmic case ($\sigma = 1$), all outcomes collapse to a single, observationally equivalent equilibrium.

This finding for the Y/K ratio has a direct and empirically relevant interpretation for the capital-output ratio (K/Y). A lower Y/K ratio implies a higher capital-to-output ratio. The values implied by our simulations (the inverse of the Y/K values in Table 3) range from approximately 2.28 to 2.44. These values are consistent with the range observed in developed economies.[35]Therefore, our model's implied ratios fall comfortably within this empirically observed range.

The stability of the steady state is determined by the eigenvalues of the Jacobian matrix derived from the linearized system of differential equations (see Section B.11). Table 3 also reports these eigenvalues. A key analytical result, detailed in Section B.11, is that one eigenvalue is given by the steady-state extraction rate, $m_4 = R/S = rs^*$. Since rs^* is strictly positive, the equilibrium is necessarily a saddle point. The numerical results in Table 3 confirm this: in every case, the system has three positive eigenvalues and only one negative (and real) eigenvalue (m_3, denoted λ_L). This implies the existence of a one-dimensional stable manifold along which the economy can converge to the steady state. For a geometric intuition of these dynamics, a 3D vector field illustrating the flow around the steady states is provided in Appendix C.9.

The analysis of the eigenvalues also reveals the influence of time preferences on the system's dynamics. The negative eigenvalue, λ_L, governs the speed of convergence along the stable saddle path. A larger absolute value implies a faster exponential decay of any deviation from the steady state.

For a low EIS ($\sigma < 1$), the economy with time-inconsistent agents exhibits a marginally faster convergence. As the table shows, $|\lambda_{L,\theta}^{\sigma<1}| > |\lambda_{L,\text{Exp}}^{\sigma<1}|$. This indicates that the trajectory approaches its steady state more rapidly in an economy with present bias. Conversely, for a high EIS ($\sigma > 1$), now the time-consistent economy (exponential) converges slightly faster, as $|\lambda_{L,\theta}^{\sigma>1}| < |\lambda_{L,\text{Exp}}^{\sigma>1}|$. In sum, the analysis reveals that time-inconsistent preferences not only shift the location and characteristics of the long-run equilibrium but also modify the dynamic path of convergence toward it.

[35]As noted by Backus et al. (2008), "[i]n the US, the capital-output ratio has been between 2.4 and 2.8 since 1970. In France, the capital-output ratio rose from 2.6 to 3.1 during the 1970s, but has been between 3.0 and 3.2 ever since 1980. In Japan, by contrast, the capital-output ratio has risen steadily, from 2 in 1970 to 3 in 1980 to almost 4 in 2004. The investment rate fell in the 1990s, but slower growth in output led to further increase in the ratio of capital to output. In Ireland, the capital-output ratio fell from 3.0 in 1986 to about 1.8 in 2005, a period of rapid growth and (apparently) low corporate tax rates."

6 Conclusions

In this paper, we study an endogenous growth model with natural resources and pollution when households have present-biased preferences. We compare an economy populated by time-consistent households with one populated by naive time-inconsistent agents.

An economy with time-inconsistent agents can, theoretically —when the EIS is greater than one—, achieve higher growth rates, allocate more workers to R&D, create ideas at a higher rate, and extract the resource less aggressively. If the EIS is equal to one, the time-inconsistent and time-consistent economies are observationally equivalent, i.e., an economy with log utility is not affected by present bias. On the contrary, if the EIS is lower than one, the economy with time-inconsistent agents will certainly experience lower growth rates. The theoretical impacts on R&D, creation of ideas, and resource extraction remain ambiguous. However, in the quantitative results, we show that such economies allocate fewer workers to R&D, generate ideas more slowly, and extract the resource more aggressively.

Hence, the generic claim that "present bias lowers growth" is not a universal law. Rather, whether present bias helps or harms long-run performance depends on which side of the knife-edge $\sigma = 1$ an economy lies. This mechanism links behavioral preferences to economic growth, resource use, innovation, and environmental dynamics in a unified, tractable framework.

This framework also allows us to quantify the costs (when EIS < 1) or benefits (when EIS > 1) of present bias in terms of economic growth, resource conservation, idea creation, and pollution. Future policy analysis should take into account interventions that may mitigate the present-bias harms when EIS < 1 (e.g., deploying certain commitment devices), while keeping the gains when EIS > 1. Thus, an institutional framework that helps present-biased agents to commit to long-term plans would increase the growth rate when the EIS is lower than one, while such commitment devices would reduce economic growth when the EIS is higher than one.

Future research could introduce heterogeneous households, with one group having time-consistent preferences and another group of fully naive time-inconsistent agents. This would allow for the development of a two-agent endogenous growth model and study how such heterogeneity affects growth rates, the creation of ideas, resource extraction and pollution.

This paper opens paths for future research by allowing present bias to be tractably incorporated into growth models.

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Online Appendix for:

Being Human: Endogenous Growth, Pollution and Natural Resources under Time-Inconsistent Preferences

Carles Mañó-Cabello

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A Main Proofs

A.1 Proof of Proposition 1

To keep the proof a bit more simple and intuitive, we show here the case in equilibrium, where the interest rate is constant over time, simplifying the analysis. For the general proof, see the online Appendix B.6.

Proof. We prove here the case for $\sigma < 1$. The study of $\sigma > 1$ is analogous. Thus, we will show that for $\sigma < 1 \iff \lambda_\theta^{ces} > \hat{\rho}$. As in equilibrium the interest is constant (see the growth rates in the BGP), we can write

$$\lambda_\theta^{ces} = \frac{\int_0^\infty \rho(j)\,\theta(j)^\sigma\, e^{-(1-\sigma)rj}\,dj}{\int_0^\infty \theta(j)^\sigma\, e^{-(1-\sigma)rj}\,dj}. \tag{A.1}$$

For convenience, define

$$I(\sigma) = \int_0^\infty \theta(j)^\sigma\, e^{-(1-\sigma)rj}\,dj,$$

$$N(\sigma) = \int_0^\infty \rho(j)\,\theta(j)^\sigma\, e^{-(1-\sigma)rj}\,dj.$$

Then,

$$\lambda_\theta^{ces}(\sigma) = \frac{N(\sigma)}{I(\sigma)}. \tag{A.2}$$

We now present the benchmark case for $\sigma = 1$, which is shown in Remark 1. This example serves as a reference for comparing the general approach used to prove the case when $\sigma \neq 1$.

Step 1. The Benchmark Case: $\sigma = 1$

When $\sigma = 1$, note that

$$I(1) = \int_0^\infty \theta(j)\,dj = \frac{1}{\hat{\rho}},$$

and

$$N(1) = \int_0^\infty \rho(j)\,\theta(j)\,dj.$$

Since by definition $\rho(j) = -\frac{\dot{\theta}(j)}{\theta(j)}$, we have $\rho(j)\,\theta(j) = -\dot{\theta}(j)$. Therefore,

$$N(1) = -\int_0^\infty \dot{\theta}(j)\,dj.$$

Applying the Fundamental Theorem of Calculus (or integration by parts) with $\theta(0) = 1$ and $\lim_{j\to\infty} \theta(j) = 0$, we obtain

$$-\int_0^\infty \dot{\theta}(j)\, dj = -\Big[\theta(j)\Big]_0^\infty = -\Big[0-1\Big] = 1.$$

Hence,

$$\lambda_\theta^{ces}(1) \;=\; \frac{1}{\int_0^\infty \theta(j)\, dj} \;=\; \hat{\rho}. \tag{A.3}$$

Step 2. The General Case: $\sigma \neq 1$

For general $\sigma \neq 1$, we have

$$N(\sigma) = \int_0^\infty \rho(j)\, \theta(j)^\sigma\, e^{-(1-\sigma)rj}\, dj.$$

Using the identity $\rho(j) = -\frac{\dot{\theta}(j)}{\theta(j)}$, we rewrite the integrand:

$$\rho(j)\, \theta(j)^\sigma = -\dot{\theta}(j)\, \theta(j)^{\sigma-1}.$$

Thus,

$$N(\sigma) = -\int_0^\infty \theta(j)^{\sigma-1}\, \dot{\theta}(j)\, e^{-(1-\sigma)rj}\, dj.$$

Now, define the function

$$F(j) \;=\; \theta(j)^\sigma\, e^{-(1-\sigma)rj}.$$

Computing its derivative using the product rule, one gets

$$F'(j) = \sigma\, \theta(j)^{\sigma-1}\dot{\theta}(j)e^{-(1-\sigma)rj} - (1-\sigma)r\, \theta(j)^\sigma e^{-(1-\sigma)rj}.$$

Rearrange this expression to solve for the term we need:

$$\sigma\, \theta(j)^{\sigma-1}\dot{\theta}(j)e^{-(1-\sigma)rj} = F'(j) + (1-\sigma)r\, F(j),$$

so that

$$\theta(j)^{\sigma-1}\dot{\theta}(j)e^{-(1-\sigma)rj} = \frac{1}{\sigma}\, F'(j) + \frac{1-\sigma}{\sigma} r\, F(j).$$

Substitute this back into the expression for $N(\sigma)$:

$$N(\sigma) = -\int_0^\infty \left\{ \frac{1}{\sigma} F'(j) + \frac{1-\sigma}{\sigma} r\, F(j) \right\} dj.$$

We split the integral:

$$N(\sigma) = -\frac{1}{\sigma}\int_0^\infty F'(j)\, dj - \frac{1-\sigma}{\sigma} r \int_0^\infty F(j)\, dj.$$

The first term is evaluated by the Fundamental Theorem of Calculus:

$$\int_0^\infty F'(j)\, dj = \lim_{b\to\infty} F(b) - F(0).$$

Our integrability assumption ensures that

$$\lim_{j\to\infty} F(j) = 0,$$

and clearly $F(0) = \theta(0)^{\sigma} e^{0} = 1$. This result is further explained at the conclusion of the proof to ensure a clear and coherent presentation here. Hence,

$$\int_0^{\infty} F'(j)\, dj = -1.$$

Thus,

$$N(\sigma) = -\frac{1}{\sigma}(-1) - \frac{1-\sigma}{\sigma} r\, I(\sigma) = \frac{1}{\sigma} - \frac{1-\sigma}{\sigma} r\, I(\sigma).$$

Substituting this into (A.2) yields

$$\lambda_{\theta}^{ces}(\sigma) \;=\; \frac{1}{\sigma\, I(\sigma)} - \frac{1-\sigma}{\sigma} r. \tag{A.4}$$

Step 3.1 Comparative Statics

Below we focus on the key claim:

$$I(\sigma) \;=\; \int_0^{\infty} \theta(j)^{\sigma} \exp\{-(1-\sigma) r\, j\}\, dj$$

is smaller (locally around $\sigma = 1$) than

$$I(1) = \int_0^{\infty} \theta(j)\, dj \quad \text{for } \sigma < 1.$$

Hence

$$\frac{1}{\sigma\, I(\sigma)} \;>\; \frac{1}{I(1)} \;=\; \hat{\rho},$$

leading to

$$\lambda_{\theta}^{ces}(\sigma) \;=\; \frac{1}{\sigma\, I(\sigma)} \;-\; \frac{1-\sigma}{\sigma} r \;>\; \hat{\rho} = \lambda_{\theta}^{ces}(1) \quad \text{for } \sigma < 1.$$

We proceed to prove the claim as follows:

Step 3.2 Definition of $I(\sigma)$ and the Idea

Recall that $I(\sigma)$ is defined by

$$I(\sigma) \;=\; \int_0^{\infty} \theta(j)^{\sigma} \exp\{-(1-\sigma) r j\}\, dj,$$

with $\theta(j) \in (0, 1]$ for $j > 0$. At $\sigma = 1$, the exponent $(1-\sigma)$ is zero, so

$$I(1) = \int_0^{\infty} \theta(j)\, dj \;= \frac{1}{\hat{\rho}}.$$

To understand how $I(\sigma)$ changes as σ moves away from 1, we look at the derivative $\frac{d}{d\sigma} I(\sigma)$ at $\sigma = 1$. If that derivative is positive, then for $\sigma < 1$ (near 1) we have $I(\sigma) < I(1)$.

Differentiating under the integral sign. Recall that

$$I(\sigma) = \int_0^\infty \underbrace{\theta(j)^\sigma \exp\{-(1-\sigma)rj\}}_{\equiv G(j,\sigma)}\, dj.$$

Under our standing assumptions on θ (positivity, monotonicity, integrability) and $r > 0$, we may apply the Leibniz rule to obtain

$$\frac{\partial}{\partial\sigma} G(j,\sigma) = \theta(j)^\sigma \ln\theta(j)\, e^{-(1-\sigma)rj} \;+\; \theta(j)^\sigma\, e^{-(1-\sigma)rj}\, rj,$$

and therefore, at $\sigma = 1$, it simplifies notably to

$$\frac{d}{d\sigma} I(\sigma)\Big|_{\sigma=1} = \int_0^\infty \theta(j)\,\big[\ln\theta(j) + rj\big]\, dj. \tag{A.5}$$

To determine the sign of the integrand, use the definition $\rho(j) = -\dot\theta(j)/\theta(j)$. Therefore, as $\frac{d}{dj}\ln\theta(j) = \frac{\dot\theta(j)}{\theta(j)}$, we have $\rho(j) = -\frac{d\ln\theta(j)}{dj}$, which is a separable differential equation. Thus,

$$\int_0^j \rho(s)ds = -\left(\ln\theta(j) - \ln\underbrace{\theta(0)}_{=1}\right) = -\left(\ln\theta(j) - 0\right) = -\ln\theta(j).$$

Thus, from $\ln\theta(j) = -\int_0^j \rho(s)\, ds$, we obtain

$$\ln\theta(j) + rj = \int_0^j \big(r - \rho(s)\big)\, ds.$$

Plugging this identity into (A.5) and applying Tonelli/Fubini (which allows us to swap the order of integration in the double integral) yields the tail–integration representation

$$\begin{aligned}\frac{d}{d\sigma} I(\sigma)\Big|_{\sigma=1} &= \int_0^\infty \theta(j)\left[\int_0^j (r-\rho(s))\, ds\right] dj = \int_0^\infty (r-\rho(s))\left[\int_s^\infty \theta(j)\, dj\right] ds \\ &= \int_0^\infty T(s)\,\big(r-\rho(s)\big)\, ds, \qquad \text{where} \qquad T(s) \;\equiv\; \int_s^\infty \theta(j)\, dj \;>\; 0. \end{aligned} \tag{A.6}$$

Sign conditions. Equation (A.6) shows that the sign of $dI/d\sigma|_{\sigma=1}$ is governed by the gap $r - \rho(s)$, averaged with positive weights $T(s)$.

If the function $\rho(j)$ is non-increasing (i.e., $\dot\rho \le 0$) and $r > \rho(0)$, then $r - \rho(s) \ge r - \rho(0) > 0$ for all $s \ge 0$, hence

$$\frac{d}{d\sigma} I(\sigma)\Big|_{\sigma=1} \;>\; 0.$$

In particular, under exponential discounting, $\rho(s) \equiv \hat\rho$ and (A.6) reduces to

$$\frac{d}{d\sigma} I(\sigma)\Big|_{\sigma=1} = (r-\hat\rho)\int_0^\infty T(s)\, ds \;>\; 0 \quad\Longleftrightarrow\quad r > \hat\rho,$$

which is equivalent to the pointwise expression $\theta(j)\big[\ln\theta(j) + rj\big] = e^{-\hat\rho j}(r-\hat\rho)\, j$, and commonly assumed in the canonical Ramsey model with exponential discounting to get positive growth rates.

Consequently, when $r > \rho(0)$ (a sufficient and, under $\dot{\rho} \leq 0$, also necessary condition for $\ln \theta(j) + rj \geq 0$ for all j), we have

$$\frac{d}{d\sigma} I(\sigma)\Big|_{\sigma=1} > 0,$$

so $I(\sigma) < I(1)$ for all $\sigma < 1$ sufficiently close to 1. Using

$$\lambda_\theta^{ces}(\sigma) = \frac{1}{\sigma\, I(\sigma)} - \frac{1-\sigma}{\sigma} r, \qquad \lambda_\theta^{ces}(1) = \frac{1}{I(1)} = \hat{\rho},$$

it follows that, locally around $\sigma = 1$,

$$\lambda_\theta^{ces}(\sigma) > \hat{\rho} \quad \text{for } \sigma < 1, \qquad \lambda_\theta^{ces}(\sigma) < \hat{\rho} \quad \text{for } \sigma > 1,$$

given that the relevant integrals converge (see the convergence result below).

Intuitively, for the model to generate positive long-run consumption growth, it is necessary for agents to have an incentive to save. This requires that the interest rate r, is sufficiently high to overcome the agent's impatience. Formally, this economic condition is captured by the inequality $\ln(\theta(j)) + rj > 0$, for all $j > 0$. This can be rewritten as $r > -\frac{\ln(\theta(j))}{j} = \frac{1}{j}\int_0^j \rho(s)ds$, which states that the interest rate must exceed the average instantaneous discount rate over any horizon.

Step 4. Discussion on the Boundary Condition

As this proof is already long enough, we will show here only the intuition and later in Lemma 2 we will derive formally the condition for σ such that the integrals converge, and $\lambda_\theta(\sigma, r)$ is well defined.

A crucial step in the above derivation is the assertion that

$$\lim_{j\to\infty} F(j) = \lim_{j\to\infty} \theta(j)^\sigma e^{-(1-\sigma)rj} = 0.$$

For $\sigma < 1$, note that $1 - \sigma > 0$, so that the exponential factor $e^{-(1-\sigma)rj}$ decays to zero, and since $\theta(j)$ is positive and decaying, the product clearly vanishes.

For $\sigma > 1$, we have $1 - \sigma < 0$, so that the term $e^{-(1-\sigma)rj} = e^{(\sigma-1)rj}$ *grows* exponentially. In order to ensure that $F(j) \to 0$, it is necessary that the decay of $\theta(j)^\sigma$ is sufficiently fast to overcome the exponential growth. For instance, if we assume that for large j the discount function is asymptotically exponential, i.e.,

$$\theta(j) \sim e^{-\kappa j} \quad \text{with } \kappa > 0,$$

then

$$\theta(j)^\sigma \sim e^{-\kappa\sigma j},$$

and

$$F(j) \sim e^{-\kappa\sigma j}\, e^{(\sigma-1)rj} = e^{-j\left(\kappa\sigma-(\sigma-1)r\right)}.$$

Thus, we require

$$\kappa\sigma - (\sigma - 1)r > 0 \quad \Longrightarrow \quad \kappa > \frac{(\sigma - 1)r}{\sigma}.$$

We assume that $\theta(j)$ is chosen (or calibrated) so that this condition holds for the range of $\sigma > 1$ under consideration. In other words, there exists a threshold value $\bar{\sigma}$ –defined later– such that for $\sigma \leq \bar{\sigma}$ the integrability condition

$$\lim_{j \to \infty} F(j) = 0$$

holds. The discount functions used in economics, such as hyperbolic or generalized hyperbolic functions, satisfy this requirement for a wide range of σ values. However, σ cannot be very high, as if it is above $\bar{\sigma}$, the integrals do not converge, which is what we see numerically as well.

Step 5. Implications for Consumption Growth

The modified Ramsey rule in our setting is given by $\frac{\dot{c}(t)}{c(t)} = \sigma\left[r - \lambda_\theta^{ces}\right]$. Thus, for positive consumption growth, it is necessary that $r > \lambda_\theta^{ces}$. In the benchmark case $\sigma = 1$, this condition reduces to $r > \hat{\rho}$. In the general case, when $\sigma < 1$ (and under positive growth, so that $r > \hat{\rho}$), our result that $\lambda_\theta^{ces} > \hat{\rho}$ implies that

$$r > \lambda_\theta^{ces} > \hat{\rho}.$$

To conclude, we have shown that under a general discount function $\theta(j)$ satisfying $\theta(0) = 1$, $\theta(j) > 0$, and $\dot{\theta}(j) < 0$ for all $j > 0$, the effective rate of time preference under CES utility,

$$\lambda_\theta^{ces}(\sigma) = \frac{1}{\sigma\, I(\sigma)} - \frac{1 - \sigma}{\sigma} r,$$

with

$$I(\sigma) = \int_0^\infty \theta(j)^\sigma e^{-(1-\sigma)rj}\, dj,$$

satisfies:

1. When $\sigma = 1$, $\lambda_\theta^{ces}(1) = \hat{\rho}$.
2. For $\sigma < 1$, we have $\lambda_\theta^{ces}(\sigma) > \hat{\rho}$.
3. For $\sigma > 1$, provided that $\theta(j)$ decays sufficiently fast (i.e., there exists $\bar{\sigma}$ such that for $\sigma \leq \bar{\sigma}$, $\lim_{j \to \infty} F(j) = 0$ holds), we obtain $\lambda_\theta^{ces}(\sigma) < \hat{\rho}$. Above that certain threshold, the corresponding integral will not converge. Thus, assuming an upper bound is in line with the finite integral under the logarithmic case ($\sigma = 1$).

Therefore, with all the subcases, we have proved the *iff* statement. Furthermore, the modified Ramsey rule requires $r > \lambda_\theta^{ces}(\sigma)$ for positive consumption growth. In particular, for $\sigma < 1$ (with $r > \hat{\rho}$) it follows that

$$r > \lambda_\theta^{ces}(\sigma) > \hat{\rho}.$$

For the sake of intuition, Figure A.1 shows graphically the consequences of this proposition.

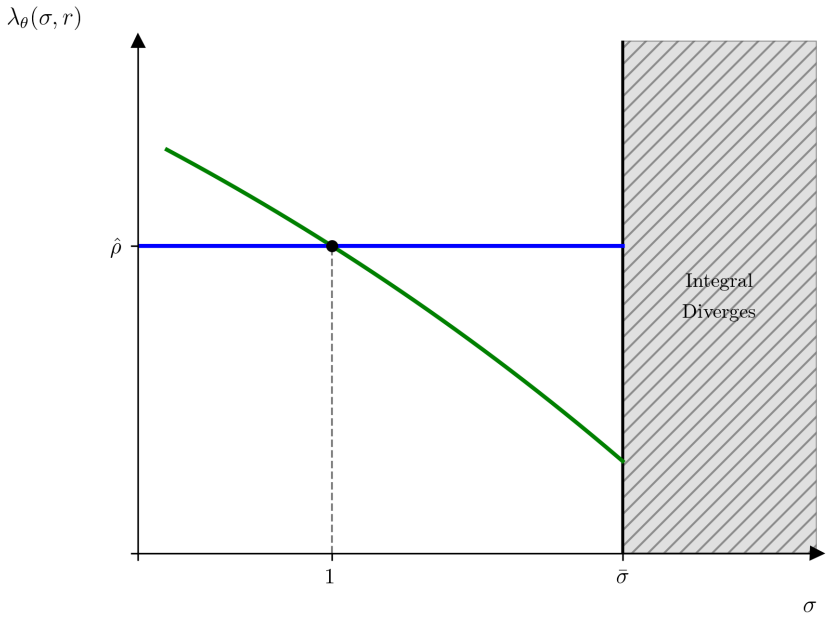


Figure A.1: Behavior of $\lambda_\theta(\sigma, r)$ showing the intersection with $\hat{\rho}$ when $\sigma = 1$, $\lambda_\theta > \hat{\rho}$ for $\sigma < 1$, and $\lambda_\theta < \hat{\rho}$ for $\sigma > 1$ with the region where the integral diverges for $\sigma > \bar{\sigma}$.

This completes the proof.

□

We now proceed to show that the EIS (or the inverse of the CRRA parameter) cannot be arbitrarily large, i.e., $\sigma < \bar{\sigma}$, which we will determine in the following Lemma. It must be below a certain threshold for the model to be well defined.

Lemma 2 (Convergence Condition for Integrals in the Effective Rate of Time Preference). *Let the discount function $\theta : [0, \infty) \to (0, 1]$ be continuously differentiable (C^1) with $\theta(0) = 1$ and $\dot{\theta}(j) < 0$ for all $j > 0$. The instantaneous discount rate is given by $\rho(j) = -\dot{\theta}(j)/\theta(j)$. To clarify the exposition and help the reader with the proof, we enumerate here the assumptions in the paper:*

1. *$\rho(j) > 0$ for all $j > 0$.*
2. *$\rho(j)$ is non-increasing, i.e., $\dot{\rho}(j) \leq 0$.*

3. *The limit* $\lim_{j\to\infty}\rho(j)=\rho_\infty$ *exists, is finite, and positive. From (2), this implies* $\rho(j)\geq\rho_\infty$ *for all* j.

4. *The interest rate* r *is positive, and crucially,* $r>\rho_\infty$. *This is to ensure that the Ramsey rule can yield positive consumption growth, which is the case we consider under exponential discounting.*

The "effective rate of time preference" $\lambda_\theta(\sigma,r)$ *is defined as* $N(\sigma;r)/I(\sigma;r)$, *where we have written them to depend explicitly on* σ *and* r, *where*

$$I(\sigma;r)=\int_0^\infty \theta(j)^\sigma\, e^{-(1-\sigma)rj}\,dj,\quad \text{and}\quad N(\sigma;r)=\int_0^\infty \rho(j)\,\theta(j)^\sigma\, e^{-(1-\sigma)rj}\,dj.$$

Then $\lambda_\theta(\sigma,r)$ *is well-defined (i.e.,* $I(\sigma;r)$ *and* $N(\sigma;r)$ *both converge to finite positive values) if* $0\leq\sigma<\bar\sigma$, *where*

$$\bar\sigma=\frac{r}{r-\rho_\infty},\tag{A.7}$$

which is strictly greater than 1, and resembles the threshold obtained in Cabo et al. (2020b), but for our framework.

Both integrals $I(\sigma;r)$ *and* $N(\sigma;r)$ *diverge if* $\sigma>\bar\sigma$. *At the boundary case* $\sigma=\bar\sigma$:

- *If* $\int_0^\infty(\rho(s)-\rho_\infty)ds$ *converges to a finite value* $K_\infty\geq 0$, *then* $I(\bar\sigma;r)$ *and* $N(\bar\sigma;r)$ *diverge. (This includes the case* $\rho(j)=\rho_\infty$ *for all* j.*)*

- *If* $\rho(j)-\rho_\infty\sim c/j$ *for some constant* $c>0$ *as* $j\to\infty$, *then* $I(\bar\sigma;r)$ *and* $N(\bar\sigma;r)$ *diverge if* $c\bar\sigma\leq 1$ *and converge if* $c\bar\sigma>1$.

Consequently, $\lambda_\theta(\sigma,r)=N(\sigma;r)/I(\sigma;r)$ *is well-defined for* $0\leq\sigma<\bar\sigma$.

Proof. The proof establishes the convergence conditions for $I(\sigma;r)$ and $N(\sigma;r)$ by analyzing the asymptotic behavior of their integrands. For brevity in the proof, we simplify and write $I(\sigma)$ and $N(\sigma)$ when r is clear from context. For the sake of clarity, we will proceed in several steps.

Step 1: Representation of $\theta(j)$ **in terms of** $\rho(j)$. The definition of the instantaneous discount rate is $\rho(j)=-\frac{d}{dj}\ln\theta(j)$. Integrating both sides from $s=0$ to $s=j$:

$$\int_0^j\rho(s)ds=-\int_0^j\frac{d}{ds}\ln\theta(s)ds=-[\ln\theta(s)]_0^j=-(\ln\theta(j)-\ln\theta(0)).$$

Given the normalization $\theta(0)=1$, we have $\ln\theta(0)=0$. Thus,

$$\ln\theta(j)=-\int_0^j\rho(s)ds.$$

Exponentiating yields the general well-known form of the discount function:

$$\theta(j) = \exp\left(-\int_0^j \rho(s)ds\right). \tag{A.8}$$

Now, let the integrand of $I(\sigma; r)$ be denoted by $h(j, \sigma, r)$, that is,

$$h(j, \sigma, r) = \theta(j)^{\sigma} e^{-(1-\sigma)rj}.$$

Since $\theta(j) \in (0, 1]$ (as $\rho(s) > 0 \implies \int_0^j \rho(s)ds > 0$ for $j > 0$) and the exponential term is positive, then, $h(j, \sigma, r) > 0$ for all $j \geq 0$ and $\sigma \geq 0$.

Step 2: Asymptotic Behavior of the Integrand $h(j, \sigma, r)$. We analyze the convergence of $I(\sigma; r) = \int_0^{\infty} h(j, \sigma, r)dj$. For an integral of a positive function, we use a common approach to the root test for series, which can be adapted: the integral converges if $\limsup_{j\to\infty}(h(j, \sigma, r))^{1/j} < 1$ and diverges if $\liminf_{j\to\infty}(h(j, \sigma, r))^{1/j} > 1$.
First, we calculate $(h(j, \sigma, r))^{1/j}$:

$$\begin{aligned}
(h(j, \sigma, r))^{1/j} &= \left(\theta(j)^{\sigma} e^{-(1-\sigma)rj}\right)^{1/j} \qquad \text{using now equation (A.8)}, \\
&= \left(\exp\left(-\sigma \int_0^j \rho(s)ds\right) \cdot \exp(-(1-\sigma)rj)\right)^{1/j} \\
&= \exp\left(-\frac{\sigma}{j} \int_0^j \rho(s)ds - (1-\sigma)r\right).
\end{aligned}$$

Let $L_{\rho} = \lim_{j\to\infty} \frac{1}{j} \int_0^j \rho(s)ds$. By Assumption 3 in this proof, $\lim_{s\to\infty} \rho(s) = \rho_{\infty}$ (a finite positive constant). Since $\rho(s)$ is continuous on $[0, \infty)$ and $\lim_{s\to\infty} \rho(s) = \rho_{\infty} > 0$, the integral $\int_0^j \rho(s)ds \to \infty$ as $j \to \infty$. Thus, the limit L_{ρ} is of the indeterminate form $\frac{\infty}{\infty}$. We can apply L'Hôpital's Rule:

$$L_{\rho} = \lim_{j\to\infty} \frac{\frac{d}{dj} \int_0^j \rho(s)ds}{\frac{d}{dj} j} = \lim_{j\to\infty} \frac{\rho(j)}{1} = \rho_{\infty},$$

where the second equality uses the Fundamental Theorem of Calculus. Therefore, $L_{\rho} = \rho_{\infty}$. We now show the limit for the Root Test Criterion. Let $L_h(\sigma, r) = \lim_{j\to\infty}(h(j, \sigma, r))^{1/j}$, and using the results above,

$$L_h(\sigma, r) = \exp\left(-\sigma\rho_{\infty} - (1-\sigma)r\right).$$

Let $E_h(\sigma, r) = -(\sigma\rho_{\infty} + (1-\sigma)r)$ represent the terms in the exponent. Then $L_h(\sigma, r) = e^{E_h(\sigma,r)}$. Therefore, it is straightforward to see that, the integral $I(\sigma; r)$ converges if $L_h(\sigma, r) < 1 \iff E_h(\sigma, r) < 0$; the integral $I(\sigma; r)$ diverges if $L_h(\sigma, r) > 1 \iff E_h(\sigma, r) > 0$; and, as it is always the case in Analysis, the test is inconclusive if $L_h(\sigma, r) = 1 \iff E_h(\sigma, r) = 0$.

Step 3: Characterizing the Threshold $\bar{\sigma}$ for $I(\sigma; r)$. Let $A(\sigma, r) = \sigma\rho_\infty + (1-\sigma)r$. The condition for convergence is $A(\sigma, r) > 0$. The condition for divergence is $A(\sigma, r) < 0$. We rewrite $A(\sigma, r)$:

$$A(\sigma, r) = \sigma\rho_\infty + r - \sigma r = r - \sigma(r - \rho_\infty).$$

We analyze the inequality when it is positive, i.e., $A(\sigma, r) > 0, \iff r - \sigma(r - \rho_\infty) > 0$. By Assumption 4, $r > \rho_\infty$, which implies $r - \rho_\infty > 0$. Dividing by $r - \rho_\infty$ (a positive quantity, so inequality direction is preserved):

$$\frac{r}{r - \rho_\infty} > \sigma \quad \text{or} \quad \sigma < \frac{r}{r - \rho_\infty}.$$

We can now define the threshold $\bar{\sigma} = \frac{r}{r-\rho_\infty}$. Since $r > 0$ and $r - \rho_\infty > 0$, we have $\bar{\sigma} > 0$. Thus:

- If $0 \leq \sigma < \bar{\sigma}$, then $A(\sigma, r) > 0 \implies E_h(\sigma, r) < 0 \implies L_h(\sigma, r) < 1$. So $I(\sigma; r)$ converges.
- If $\sigma > \bar{\sigma}$, then $A(\sigma, r) < 0 \implies E_h(\sigma, r) > 0 \implies L_h(\sigma, r) > 1$. So $I(\sigma; r)$ diverges.

Step 4: Behavior of $I(\sigma; r)$ at the Boundary $\sigma = \bar{\sigma}$. Although economically speaking this case it not extremely relevant, for completeness and formality, we now discuss the boundary case. At $\sigma = \bar{\sigma}$, we have $A(\bar{\sigma}, r) = r - \bar{\sigma}(r - \rho_\infty) = r - \frac{r}{r-\rho_\infty}(r - \rho_\infty) = 0$. This implies $E_h(\bar{\sigma}, r) = 0$, so $L_h(\bar{\sigma}, r) = e^0 = 1$. The root test criterion is inconclusive. Therefore, a more refined analysis of $h(j, \bar{\sigma}, r)$ is required. We then use the log of function h at that critical threshold,

$$\ln h(j, \bar{\sigma}, r) = \bar{\sigma} \ln \theta(j) - (1 - \bar{\sigma})rj = -\bar{\sigma} \int_0^j \rho(s)ds - (1 - \bar{\sigma})rj.$$

Using $\rho(s) = \rho_\infty + (\rho(s) - \rho_\infty)$:

$$\begin{aligned}
\ln h(j, \bar{\sigma}, r) &= -\bar{\sigma} \int_0^j (\rho_\infty + (\rho(s) - \rho_\infty))ds - (1 - \bar{\sigma})rj \\
&= -\bar{\sigma}\rho_\infty j - \bar{\sigma} \int_0^j (\rho(s) - \rho_\infty)ds - rj + \bar{\sigma}rj \\
&= -\underbrace{(\bar{\sigma}\rho_\infty + (1 - \bar{\sigma})r)}_{=A(\bar{\sigma}, r)=0} j - \bar{\sigma} \int_0^j (\rho(s) - \rho_\infty)ds \\
&= -\bar{\sigma} \int_0^j (\rho(s) - \rho_\infty)ds.
\end{aligned}$$

So, $h(j, \bar{\sigma}, r) = \exp\left(-\bar{\sigma} \int_0^j (\rho(s) - \rho_\infty)ds\right)$. We now define the auxiliary function $K(j) = \int_0^j (\rho(s) - \rho_\infty)ds$. Since $\rho(s) \geq \rho_\infty$ (Assumption 3), $K(j)$ is non-decreasing. We now consider the different cases. *Case 4a:* $\lim_{j\to\infty} K(j) = K_\infty$ *(finite)*: This occurs if $\rho(s) - \rho_\infty$ decays sufficiently fast (e.g., faster than $1/j$). Then $\lim_{j\to\infty} h(j, \bar{\sigma}, r) = \exp(-\bar{\sigma}K_\infty)$. If $K_\infty > 0$ or if $K_\infty = 0$ (which implies $\rho(s) = \rho_\infty$ for all s), then $\lim_{j\to\infty} h(j, \bar{\sigma}, r)$ is a positive constant (or 1). In this case, $\int_0^\infty h(j, \bar{\sigma}, r)dj$ diverges. For instance, if $\rho(s) = \rho_\infty$ for all s, then $\theta(j) = e^{-\rho_\infty j}$, which is the case under the exponential discount

function. The integral $I(\bar{\sigma};r)$ becomes $\int_0^\infty e^{-\rho_\infty \bar{\sigma} j} e^{-(1-\bar{\sigma})rj} dj = \int_0^\infty e^{-A(\bar{\sigma},r)j} dj = \int_0^\infty e^0 dj = \int_0^\infty 1 dj$, which diverges.

Case 4b: $\lim_{j\to\infty} K(j) = \infty$: This occurs if $\rho(s) - \rho_\infty$ decays slowly (e.g., like $1/j$ or slower).

- If $\rho(s) - \rho_\infty \sim c/s$ for large s ($c > 0$): Then $K(j) \sim c \ln j$. So, $h(j,\bar{\sigma},r) \sim \exp(-\bar{\sigma} c \ln j) = j^{-c\bar{\sigma}}$. The integral $\int_1^\infty j^{-c\bar{\sigma}} dj$ converges if $c\bar{\sigma} > 1$ and diverges if $c\bar{\sigma} \leq 1$.
- If $\rho(s) - \rho_\infty$ decays slower than $1/s$ but still tends to 0 (e.g., $\rho(s) - \rho_\infty \sim 1/(j \ln j)$): Then $K(j)$ grows slower than $\ln j$ (e.g., $K(j) \sim \ln(\ln j)$). In this case, $h(j,\bar{\sigma},r)$ decays slower than j^{-p} for any $p > 0$ small enough, and $\int_0^\infty h(j,\bar{\sigma},r) dj$ would diverge.

For many common economic discount functions (e.g., generalized hyperbolic), the decay falls into Case 4a or Case 4b with $c\bar{\sigma} \leq 1$, leading to divergence at $\sigma = \bar{\sigma}$.

Finally, we have shown that $I(\sigma;r)$ converges for $0 \leq \sigma < \bar{\sigma}$, diverges for $\sigma > \bar{\sigma}$, and at $\sigma = \bar{\sigma}$, divergence occurs if $\int_0^\infty (\rho(s) - \rho_\infty) ds < \infty$ or if $\rho(s) - \rho_\infty \sim c/s$ with $c\bar{\sigma} \leq 1$. Convergence is possible if $\rho(s) - \rho_\infty \sim c/s$ with $c\bar{\sigma} > 1$.

Since $h(j,\sigma,r) > 0$, if $I(\sigma;r)$ converges, its value is strictly positive.

Step 5: Convergence of $N(\sigma;r)$. The integrand for $N(\sigma;r)$ is $\rho(j)h(j,\sigma,r)$. Since $\rho(j) > 0$ (Assumption 1) and $h(j,\sigma,r) > 0$, this integrand is also positive. We use the Limit Comparison Test. Consider the ratio of the integrand of $N(\sigma;r)$ to the integrand of $I(\sigma;r)$:

$$\lim_{j\to\infty} \frac{\rho(j)h(j,\sigma,r)}{h(j,\sigma,r)} = \lim_{j\to\infty} \rho(j) = \rho_\infty.$$

Since ρ_∞ is a positive finite constant (Assumption 3), the integrals $N(\sigma;r) = \int_0^\infty \rho(j)h(j,\sigma,r) dj$ and $I(\sigma;r) = \int_0^\infty h(j,\sigma,r) dj$ have the same convergence behavior. Therefore:

- $N(\sigma;r)$ converges if $I(\sigma;r)$ converges (i.e., for $0 \leq \sigma < \bar{\sigma}$).
- $N(\sigma;r)$ diverges if $I(\sigma;r)$ diverges (i.e., for $\sigma > \bar{\sigma}$, and under the same conditions as $I(\sigma;r)$ at $\sigma = \bar{\sigma}$).

Since $\rho(j) > 0$ and $h(j,\sigma,r) > 0$, if $N(\sigma;r)$ converges, its value is strictly positive.

Finally, we have shown that both integrals $I(\sigma;r)$ and $N(\sigma;r)$ converge to positive finite values if $0 \leq \sigma < \bar{\sigma} = \frac{r}{r-\rho_\infty}$. Both integrals diverge if $\sigma > \bar{\sigma}$. At the boundary $\sigma = \bar{\sigma}$, the convergence/divergence behavior of both integrals depends on the rate at which $\rho(j)$ approaches ρ_∞, as detailed in Step 4. Consequently, the ratio $\lambda_\theta(\sigma,r) = N(\sigma;r)/I(\sigma;r)$, interpreted as an effective rate of time preference, is well-defined for $0 \leq \sigma < \bar{\sigma}$.

This completes the proof for the threshold $\bar{\sigma}$ and the convergence conditions for the integrals in the effective rate of time preference. □

A.2 Proposition 2. The equilibrium conditions along the BGP

From the FOC of the intermediate monopolist (B.6) and final production (27) notice that we can write

$$r = (1-\beta_1-\beta_2)^2 \frac{Y(s)}{K(s)}, \tag{A.9}$$

which will be constant over time.[36] Consequently, the modified Ramsey rule (23), can be rewritten as

$$\dot{c}(s) = c(s)\,\sigma \left[(1-\beta_1-\beta_2)^2 \frac{Y(s)}{K(s)} - \lambda_\theta^{ces} \right]. \tag{A.10}$$

From the definition of aggregate capital accumulation

$$\dot{K}(s) = \frac{1}{1-\beta_1-\beta_2} K(s)^{1-\beta_1-\beta_2} \left[N(s)L_F(s)\right]^{\beta_1} \left[N(s)R(s)\right]^{\beta_2} - c(s), \tag{A.11}$$

and from the previous Ramsey rule, we get,

$$\gamma_{\frac{c}{K}}^{ces} = \frac{Y(s)}{K(s)} \left[\sigma(1-\beta_1-\beta_2)^2 - 1\right] - \sigma\lambda_\theta^{ces} + \frac{c(s)}{K(s)}. \tag{A.12}$$

Noting that the law of motion of the innovation possibilities frontier, γ_N, is given by eq. (14), we can derive a law of motion for the extraction of the natural resource, γ_R. Plugging the growth rate of the finial production function (27) into (28) gives,

$$(1-\beta_1-\beta_2)\gamma_K + (\beta_1+\beta_2)\gamma_N + \beta_1\gamma_{L_F} - (1-\beta_2)\gamma_R = (1-\beta_1-\beta_2)^2 \frac{Y(s)}{K(s)}. \tag{A.13}$$

Substituting the law of motion of aggregate capital eq. (A.11) and the law of motion of the innovation possibilities frontier eq. (14) into the previous expression (A.13), we get,

$$\begin{aligned}(1-\beta_2)\gamma_R &= (1-\beta_1-\beta_2)(\beta_1+\beta_2)\frac{Y(s)}{K(s)} \\ &-(1-\beta_1-\beta_2)\frac{c(s)}{K(s)} + \eta(\beta_1+\beta_2)\left(1-L_F(s)\right) + \beta_1\gamma_{L_F}.\end{aligned} \tag{A.14}$$

[36] It can be clearly seen by taking logs and differentiating with respect to time, and as in the steady state $\gamma_Y = \gamma_K$, implies that the growth rate of $r(s)$ is constant, thus, $\gamma_r = 0 \Rightarrow r \in (0,1)$.

Now, in order to get the law of motion of the labor used in the final sector, γ_{L_F}, we proceed as follows. First, from the FOC of the final producer with respect to labor eq. (7) and the equilibrium condition from the R&D sector eq. (15) leads to

$$p_{RD}(s) = \beta_1 \frac{Y(s)}{\eta N(s) L_F(s)}. \tag{A.15}$$

Taking logs and differentiating with respect to time one obtains,

$$\gamma_{p_{RD}} = \gamma_Y - \gamma_N - \gamma_{L_F}. \tag{A.16}$$

Now, plugging the free entry conditions of the representative firm of the intermediate-good sector (B.9), eq. (A.9), the growth rate of the net final production, and the law of motion of the innovation possibilities frontier (14), into the previous expression (A.16) gives

$$\begin{aligned}(1-\beta_1-\beta_2)^2 \frac{Y(s)}{K(s)} - \frac{\pi(\nu,s)}{p_{RD}(s)} = \\ (1-\beta_1-\beta_2)\gamma_K - (1-\beta_1-\beta_2)\eta(1-L_F(s)) - (1-\beta_1)\gamma_{L_F} + \beta_2\gamma_R.\end{aligned} \tag{A.17}$$

From the demand for machine $\nu \in [0, N(s)]$, eq. (6), the profits of the monopolist producing input ν, eq. (B.7), and using eq. (27), we get

$$\pi(\nu,s) = (1-\beta_1-\beta_2)(\beta_1+\beta_2)\frac{Y(s)}{N(s)}. \tag{A.18}$$

Note that the monopolist's profits, $\pi(\nu,s)$, are influenced by climate change. Specifically, $Y(s)$, which represents final output net of damages, decreases as global temperatures rise. Dividing now the previous profits expression by eq. (A.15), the prices charged by the representative firm of the R&D sector $p_{RD}(s)$, i.e., price of a patent, we get

$$\frac{\pi(\nu,s)}{p_{RD}(s)} = \eta L_F(s) \frac{(1-\beta_1-\beta_2)(\beta_1+\beta_2)}{\beta_1}. \tag{A.19}$$

Substituting the growth rate of aggregate capital in (A.11) and (A.19) into (A.17) results in

$$\begin{aligned}(1-\beta_1)\gamma_{L_F} = (1-\beta_1-\beta_2)(\beta_1+\beta_2)\frac{Y(s)}{K(s)} - (1-\beta_1-\beta_2)\frac{c(s)}{K(s)} \\ - (1-\beta_1-\beta_2)\eta(1-L_F(s)) + \beta_2\gamma_R + \eta L_F(s)\frac{(1-\beta_1-\beta_2)(\beta_1+\beta_2)}{\beta_1}.\end{aligned} \tag{A.20}$$

We can now write a system of equations formed by (A.14) and (A.20) that will define the growth rate of the extraction of the resources, γ_R, and the growth rate of the labor used to produce the final good γ_{L_F}, eq. (33):

$$\gamma_R = (\beta_1+\beta_2)\frac{Y(s)}{K(s)} - \frac{c(s)}{K(s)} + \eta(1-L_F(s))\frac{\beta_2}{1-\beta_1-\beta_2} + (\beta_1+\beta_2)\eta L_F(s), \tag{A.21}$$

$$\gamma_{L_F} = (\beta_1+\beta_2)\frac{Y(s)}{K(s)} - \frac{c(s)}{K(s)} + \eta(1-L_F(s))\left[\frac{\beta_2-(1-\beta_1-\beta_2)}{1-\beta_1-\beta_2}\right] + \eta L_F(s)\frac{(\beta_1+\beta_2)(1-\beta_2)}{\beta_1}. \tag{A.22}$$

In order to get the growth rate of output (net of damages) per capital, $\gamma_{\frac{Y}{K}}$ we proceed as follows. From the final production (27), the law of motion of the aggregate capital (A.11), the dynamics of the innovation possibilities frontier (14), the growth rate of the extraction of the natural resources (A.21) and the growth rate of the labor used in the final good production (A.22), one can derive an expression for $\gamma_{\frac{Y}{K}}$. Dividing (27) by $K(s)$, taking logs and differentiating with respect to time s gives

$$\gamma_{\frac{Y}{K}} = (\beta_1+\beta_2)\gamma_N - (\beta_1+\beta_2)\gamma_K + \beta_1\gamma_{L_F} + \beta_2\gamma_R. \tag{A.23}$$

Plugging in equations (A.11), (14), (A.21) and (A.22) into the previous expression (A.23), we finally get

$$\gamma_{\frac{Y}{K}} = (\beta_1+\beta_2)\eta L_F(S) - (1-\beta_1-\beta_2)(\beta_1+\beta_2)\frac{Y(s)}{K(s)} + \frac{\beta_2}{1-\beta_1-\beta_2}\eta(1-L_F(s)). \tag{A.24}$$

Finally, in order to derive the dynamics of pollution, we start from its definition eq. (16), together with the final production (26) and get

$$P_\theta(s) = \left[\frac{1}{1-\beta_1-\beta_2}\left(\frac{r}{1-\beta_1-\beta_2}\right)^{-\frac{1-\beta_1-\beta_2}{\beta_1+\beta_2}}\right]^{\mu_2} \cdot N_\theta(s)^{\mu_2} \cdot L_F^\theta(s)^{\mu_2\frac{\beta_1}{\beta_1+\beta_2}} \cdot R_\theta(s)^{\left(\mu_1+\mu_2\frac{\beta_2}{\beta_1+\beta_2}\right)}. \tag{A.25}$$

Note that all the variables are dependent on the discount function, that is, whether agents are time-consistent or not. The growth rate of pollution is given in the following corollary.

Corollary 7. *The growth rate of pollution along the BGP is,*

$$\gamma_P^\theta(s) = \mu_2 \cdot \gamma_N^\theta(s) + \left(\mu_1 + \mu_2\frac{\beta_2}{\beta_1+\beta_2}\right) \cdot \gamma_R^\theta(s). \tag{A.26}$$

Proof. It is straightforward by taking logs and differentiating expression (A.25) with respect to time s along the BGP, where $\gamma_{L_F}^\theta(s) = 0$. □

One can observe that theoretically, pollution could increase, stay constant or decrease over time.

A.3 Proposition 3. Properties of the Steady State

In the steady state, taking logs and differentiation with respect to time for the final production (27) gives,

$$\gamma^{\theta} = \gamma_N^{\theta} + \frac{\beta_2}{\beta_1 + \beta_2}\gamma_R^{\theta}. \tag{A.27}$$

Making use of the modified Euler equation (A.10), which gives the growth rate of consumption, and equation (28), which shows the differences in growth rates between the final output and the extraction of the resource, gives the growth rate of the extraction of the natural resource

$$\gamma_R^{\theta} = \gamma^{\theta}\left(\frac{\sigma - 1}{\sigma}\right) - \lambda_{\theta}. \tag{A.28}$$

We first define $A_0 = \frac{(1-\beta_1-\beta_2)(\beta_1+\beta_2)}{\beta_1} \in (0, \infty)$, due to the constraints $\{0 < \beta_1 < 1\} \wedge \{0 < \beta_2 < 1\} \wedge \{0 < \beta_1 + \beta_2 < 1\}$, and $H_2 = \frac{\sigma}{\beta_1 + A_0(\sigma\beta_1+\beta_2)+\beta_2(2-\sigma)}$, is assumed to be positive. The reason for H_2 to be positive is that, if we recover exponential discounting, the economy has a positive growth rate. Plugging now (A.28) into (A.27) gives

$$\gamma^{\theta} = \left[(\beta_1 + \beta_2)\gamma_N^{\theta} - \beta_2\lambda_{\theta}\right]\left[\frac{\sigma}{\sigma\beta_1 + \beta_2}\right]. \tag{A.29}$$

Using the dynamics of the innovation possibilities frontier (14), and noting that $L_{R\&D}(s) = 1 - L_F(s)$, it follows that in the steady state[37]

$$\gamma_N^{\theta} - \eta = -\eta L_F^{\theta,*}. \tag{A.30}$$

Using now the growth rates of the extraction of the resource (32) and the growth rate of the labor used in the final output production (33), together with the fact that in the steady state $\gamma_{L_F} = 0$, one gets,

$$\gamma_R^{\theta} = \eta - (1 + A_0)\eta \cdot L_F^{\theta,*}. \tag{A.31}$$

Using the previous expression together with (A.28) gives the value of the labor force used in the final sector in the steady state

$$L_F^{\theta,*} = \frac{\eta + \lambda_{\theta} - \left(\frac{\sigma-1}{\sigma}\right)\gamma^{\theta}}{(1 + A_0)\eta}. \tag{A.32}$$

Plugging now this expression (A.32) into (A.30), leads to the growth rate of the new intermediate machines,

[37] Observe that in the steady state, labor used in the final sector and the R&D sector will be stationary. This is why we will use the * notation.

$$\gamma_N^\theta = \eta \frac{A_0}{1+A_0} - \frac{\lambda_\theta - \left(\frac{\sigma-1}{\sigma}\right)\gamma^\theta}{1+A_0}. \tag{A.33}$$

In order to get the growth rate of the economy, we substitute (A.28) and (A.33) into (A.27), which gives

$$\gamma^\theta = H_2 \times \left\{\eta \cdot (\beta_1+\beta_2) A_0 - \lambda_\theta \cdot [\beta_1+\beta_2+\beta_2(1+A_0)]\right\}. \tag{A.34}$$

Finally, from (35), noticing that in the steady state labor in the final sector is constant, pollution evolves according to

$$\gamma_P^\theta(s) = \mu_2 \cdot \eta(1-L_F^{\theta,*}) + \left(\mu_1 + \mu_2 \frac{\beta_2}{\beta_1+\beta_2}\right)\gamma_R^\theta. \tag{A.35}$$

Finally, regarding the positivity of H_2, which is the normal case for experiencing porisitve growth rate in the basic endogenous growth model, certain conditions must hold,

Remark 3 (Sign of H_2). *The denominator of H_2 equals $(\beta_1 + A_0\beta_2 + 2\beta_2) + \sigma(A_0\beta_1 - \beta_2)$. Thus $H_2 > 0$ for all $\sigma > 0$ if $A_0\beta_1 \geq \beta_2$; otherwise $H_2 > 0$ for $\sigma \in (0, \sigma_H)$ with $\sigma_H = \frac{\beta_1+A_0\beta_2+2\beta_2}{\beta_2-A_0\beta_1}$.*

This completes the proof in Proposition 3. □

A.4 Lemma 1. Evolution of the exhaustible resource, extraction & the extraction rate

The growth rate of the extraction of the resource is given by (38), which is constant. One can solve the corresponding ODE, which gives the following expression for the extraction of the resource:

$$R^\theta(s) = R_0 \exp\left\{\gamma_R^\theta \cdot s\right\}. \tag{A.36}$$

Having the evolution of the extraction strategy $R(s)$, we can plug it into the dynamics of the non-renewable resource (1) and solving the corresponding differential equation to get the evolution of the natural resource, with an initial amount of natural resource of $S(0) = S_0$ given, leads to

$$S^\theta(s) = \underbrace{\left(\frac{\gamma_R^\theta \cdot S_0 + R_0^\theta}{\gamma_R^\theta}\right)}_{\text{constant}} - \frac{R_0^\theta}{\gamma_R^\theta} e^{(\gamma_R^\theta \cdot s)}. \tag{A.37}$$

In order to fulfill the transversality condition (TVC) of the natural resource, i.e., that the resource will be all exhausted over the whole time horizon (see Section 2.1), $\lim_{s\to\infty} S(s) = 0$, one gets the condition

$$R_0^\theta = \underbrace{-\gamma_R^\theta \cdot S_0}_{>0}, \tag{A.38}$$

which is thus dependent on the discount function θ.[38] Consequently, the exhaustible resource evolves as,

$$S^\theta(s) = S_0 \exp\left\{\gamma_R^\theta \cdot s\right\}, \tag{A.39}$$

and the extraction rate $R^\theta(s)/S^\theta(s)$ is

$$\frac{R^\theta(s)}{S^\theta(s)} = -\underbrace{\gamma_R^\theta}_{<0} > 0, \tag{A.40}$$

which is constant over time. This proofs the result. □

A.5 Theorem 1. Derivation of the Impact of Time-Inconsistency on Steady-State Economic Growth, R&D Labor Force, Extraction of the Resource, Creation of Ideas and Pollution

We now provide a detailed derivation of Theorem 1, which shows the impact of a shift from time consistency to time-inconsistency on the steady-state growth rates and important outcomes. We show how the effective discount parameter, λ_θ, influences key steady-state variables of the model presented in Section B.11 and Proposition 3. Note that time consistency was given by $\lambda_\theta = \hat{\rho}$ (a constant rate of time preference) and time-inconsistency by $\lambda_\theta = \lambda_\theta^{ces}$. The relationship between λ_θ^{ces} and $\hat{\rho}$ is governed by σ (the elasticity of intertemporal substitution), as stated in Proposition 1.

Here we just write again the relevant equations of the steady-state equations from Proposition 3 for helping the reasder follow the argument.

- Overall economic growth rate (given by eq (36)):

$$\gamma^\theta = H_2 \left\{\eta\left(\beta_1 + \beta_2\right) A_0 - \lambda_\theta\left[\beta_1 + \beta_2 + \beta_2(1 + A_0)\right]\right\}$$

- Labor in final good sector (given by eq (37)):

$$L_F^{\theta,*} = \frac{\eta + \lambda_\theta - \left(\frac{\sigma-1}{\sigma}\right)\gamma^\theta}{(1 + A_0)\eta}$$

- Growth rate of resource extraction (given by eq (38)):

$$\gamma_R^\theta = \eta - (1 + A_0)\eta \cdot L_F^{\theta,*}$$

[38] Note that the initial amount of the resource is identical for all agents, regardless of how they discount the future. However, the initial extraction rate does depend on the discount function because of general-equilibrium effects.

- Growth rate of new ideas (given by eq (39)):

$$\gamma_N^\theta = \frac{\eta A_0}{1+A_0} - \frac{1}{1+A_0}\left(\lambda_\theta - \left(\frac{\sigma-1}{\sigma}\right)\gamma^\theta\right)$$

- Growth rate of pollution (given by eq (40)):

$$\gamma_P^\theta = \mu_2\eta(1-L_F^{\theta,*}) + \left(\mu_1 + \mu_2\frac{\beta_2}{\beta_1+\beta_2}\right)\gamma_R^\theta$$

We define $\mathcal{K}_\lambda \equiv [\beta_1+\beta_2+\beta_2(1+A_0)]$. From the problem context, $\beta_1, \beta_2 \in (0,1)$ with $\beta_1+\beta_2 < 1$, and $A_0 = \frac{(1-\beta_1-\beta_2)(\beta_1+\beta_2)}{\beta_1} > 0$. Thus, $\mathcal{K}_\lambda > 0$. The parameter $H_2 = \frac{\sigma}{\beta_1+A_0(\sigma\beta_1+\beta_2)+\beta_2(2-\sigma)}$ is positive (see Remark 3). and ensures positive growth rate.

We analyze the partial derivatives of the key variables with respect to λ_θ.

A.5.1 Sensitivity of the Overall Economic Growth Rate (γ^θ)

The partial derivative with respect to λ_θ is:

$$\frac{\partial\gamma^\theta}{\partial\lambda_\theta} = -H_2\mathcal{K}_\lambda < 0$$

which is negative, given that $H_2 > 0$ and $\mathcal{K}_\lambda > 0$. This implies that an increase in the effective discount parameter λ_θ (reflecting greater impatience or inconsistency) reduces the overall steady-state economic growth rate.

Thus, as for $\sigma > 1$, λ_θ is smaller for time-inconsistent agents than for time-consistent agents ($\lambda_\theta < \hat{\rho}$), meaning that inconsistent agents experience higher growth rates, as it would be a reduction in λ_θ.

Conversely, for $\sigma < 1$, it implies an increase in λ_θ, and thus, time-inconsistent agents experience lower growth rates, as the derivative above is negative.

A.5.2 Sensitivity of Labor in Final Good Sector ($L_F^{\theta,*}$)

The partial derivative with respect to λ_θ is:

$$\frac{\partial L_F^{\theta,*}}{\partial\lambda_\theta} = \frac{1}{(1+A_0)\eta}\left[1-\left(\frac{\sigma-1}{\sigma}\right)\frac{\partial\gamma^\theta}{\partial\lambda_\theta}\right]$$

Substituting $\frac{\partial\gamma^\theta}{\partial\lambda_\theta} = -H_2\mathcal{K}_\lambda$, one gets:

$$\frac{\partial L_F^{\theta,*}}{\partial\lambda_\theta} = \frac{1}{(1+A_0)\eta}\left[1+\left(\frac{\sigma-1}{\sigma}\right)H_2\mathcal{K}_\lambda\right]$$

We analyze the sign of this derivative based on the value of σ:

Case $\sigma > 1$: In this case, $\frac{\sigma-1}{\sigma} > 0$. Since $H_2 > 0$ and $\mathcal{K}_\lambda > 0$, the term $\left(\frac{\sigma-1}{\sigma}\right) H_2 \mathcal{K}_\lambda$ is positive. Therefore, $1 + \left(\frac{\sigma-1}{\sigma}\right) H_2 \mathcal{K}_\lambda > 1 > 0$. Thus, $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta} > 0$ when $\sigma > 1$. Therefore, as moving from TC to TI agents whe $\sigma > 1$ is a reduction in λ_θ, the labor in the final good sector decreases, which means that more labor is allocated to the R&D sector. This can be seen as well in the numerical simulations in Section 5, corroborating the analytical results.

Case $\sigma < 1$: In this case, a move from TC to TI agents implies an increase in λ_θ, as $\lambda_\theta^{ces} > \hat{\rho}$. Now $\frac{\sigma-1}{\sigma} < 0$. We can write $\frac{\sigma-1}{\sigma} = -\left(\frac{1-\sigma}{\sigma}\right)$, where $\frac{1-\sigma}{\sigma} > 0$. The derivative becomes:

$$\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta} = \frac{1}{(1+A_0)\eta}\left[1 - \left(\frac{1-\sigma}{\sigma}\right) H_2 \mathcal{K}_\lambda\right]$$

Let $C_1 \equiv \left(\frac{1-\sigma}{\sigma}\right) H_2 \mathcal{K}_\lambda$. Note that $C_1 > 0$. The sign of $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta}$ depends on whether C_1 is greater than, equal to, or less than 1:

- If $C_1 < 1$, then $1 - C_1 > 0$, so $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta} > 0$, which now means that a move from TC to TI households implies an increase in $L_F^{\theta,*}$, which means that less labor is allocated to the R&D sector.
- If $C_1 > 1$, then $1 - C_1 < 0$, so $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta} < 0$, meaning less labor is allocated to the final good sector, and more labor is allocated to the R&D sector.
- If $C_1 = 1$, then $1 - C_1 = 0$, so $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta} = 0$, which shoes no effect on the allocation of labor between sectors when moving from TC to TI agents.

Case $\sigma = 1$ (Logarithmic Utility): Note however that under the logarithmic case, as shown in Proposition 1, $\lambda_\theta^{ces} = \hat{\rho}$ when $\sigma = 1$, then λ_θ does not change between time-consistent and time-inconsistent agents, implying no change in $L_F^{\theta,*}$ via this channel. This is a very particular case, showing that it is irrelevant if agents have tendecies to procrastinate or not, if they have log utility.

A.5.3 Sensitivity of Resource Extraction Growth Rate (γ_R^θ)

The partial derivative with respect to λ_θ is:

$$\frac{\partial \gamma_R^\theta}{\partial \lambda_\theta} = -(1+A_0)\eta \frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta}$$

This shows that $\frac{\partial \gamma_R^\theta}{\partial \lambda_\theta}$ has the opposite sign to $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta}$. Thus, for $\sigma > 1$, $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta}$ is positive, and the derivative above is negative. Under the $\sigma > 1$ scenario, moving from TC to TI means a reduction in λ_θ, thus implying a "less aggressive" growth rate of the extraction of the resource. As γ_R^θ is negative, it means γ_R^θ will increase toward zero (i.e., its absolute value shrinks), thereby attenuating the downward pressure on extraction. This can be seen numerically in Section 5 (see Figure C.6 and Table 2). For the case $\sigma < 1$, one considers all the subcases above depending on C_1, so it could go either way.

A.5.4 Sensitivity of the Growth Rate of New Ideas (γ_N^θ)

The partial derivative with respect to λ_θ is:

$$\frac{\partial \gamma_N^\theta}{\partial \lambda_\theta} = -\frac{1}{1+A_0}\left(1-\left(\frac{\sigma-1}{\sigma}\right)\frac{\partial \gamma^\theta}{\partial \lambda_\theta}\right)$$

Substitute $\frac{\partial \gamma^\theta}{\partial \lambda_\theta} = -H_2\mathcal{K}_\lambda$:

$$\frac{\partial \gamma_N^\theta}{\partial \lambda_\theta} = -\frac{1}{1+A_0}\left(1-\left(\frac{\sigma-1}{\sigma}\right)(-H_2\mathcal{K}_\lambda)\right)$$
$$\frac{\partial \gamma_N^\theta}{\partial \lambda_\theta} = -\frac{1}{1+A_0}\left(1+\left(\frac{\sigma-1}{\sigma}\right)H_2\mathcal{K}_\lambda\right)$$

We analyze the sign of this derivative based on the value of σ:

Case $\sigma > 1$: In this case, $\frac{\sigma-1}{\sigma} > 0$. Since $H_2 > 0$ and $\mathcal{K}_\lambda > 0$, the term $\left(\frac{\sigma-1}{\sigma}\right)H_2\mathcal{K}_\lambda$ is positive. Therefore, $1+\left(\frac{\sigma-1}{\sigma}\right)H_2\mathcal{K}_\lambda > 1 > 0$. Thus, $\frac{\partial \gamma_N^\theta}{\partial \lambda_\theta} = -\frac{1}{1+A_0} \times (\text{positive quantity}) < 0$. As for $\sigma > 1$ moving from being time-consistent to time-inconsistent actually means reducing λ_θ, it implies that the creation of ideas is higher for time-inconsistent agents. See also the results in the numerical simulations (Table 2).

Case $\sigma < 1$: In this case, $\frac{\sigma-1}{\sigma} < 0$. We write $\frac{\sigma-1}{\sigma} = -\left(\frac{1-\sigma}{\sigma}\right)$, where $\frac{1-\sigma}{\sigma} > 0$. The derivative becomes:

$$\frac{\partial \gamma_N^\theta}{\partial \lambda_\theta} = -\frac{1}{1+A_0}\left(1-\left(\frac{1-\sigma}{\sigma}\right)H_2\mathcal{K}_\lambda\right)$$

Let $C_1 \equiv \left(\frac{1-\sigma}{\sigma}\right)H_2\mathcal{K}_\lambda$ (as defined in Section A.5.2). We know $C_1 > 0$. The sign of $\frac{\partial \gamma_N^\theta}{\partial \lambda_\theta}$ depends on $1-C_1$:

- If $C_1 < 1$, then $1-C_1 > 0$, so $\frac{\partial \gamma_N^\theta}{\partial \lambda_\theta} < 0$.As with $\sigma < 1$ moving from time consistency to time-inconsistency means increasing λ_θ, it implies that the creation of ideas is lower for time-inconsistent agents.
- If $C_1 > 1$, then $1-C_1 < 0$, so $\frac{\partial \gamma_N^\theta}{\partial \lambda_\theta} > 0$. The creation of ideas is higher for time-inconsistent agents. This is actually the case shown in the numerical simulations (Table 2).
- If $C_1 = 1$, then $1-C_1 = 0$, so $\frac{\partial \gamma_N^\theta}{\partial \lambda_\theta} = 0$. The creation of ideas is the same for both the economy populated by time-consistent households than for time-inconsistent households.

Case $\sigma = 1$ (Logarithmic Utility): As $\lambda_\theta^{ces} = \hat{\rho}$ for $\sigma = 1$, the change $\Delta\lambda_\theta = \lambda_\theta^{ces} - \hat{\rho} = 0$. Thus, the first-order effect on γ_N^θ from moving to time-inconsistency is zero: $\Delta\gamma_N^\theta \approx \frac{\partial \gamma_N^\theta}{\partial \lambda_\theta} \times 0 = 0$.

Thus, for $\sigma < 1$, the growth rate of the creation of ideas could decrease if we move from an economy with time-consistent households to an economy with time-inconsistent households, or it could increase, depending on the value of C_1.

A.5.5 Sensitivity of Pollution Growth Rate (γ_P^θ) to λ_θ

Let $\tilde{\mu} \equiv \mu_1 + \mu_2 \frac{\beta_2}{\beta_1+\beta_2}$. Then $\gamma_P^\theta = \mu_2\eta - \mu_2\eta L_F^{\theta,*} + \tilde{\mu}\gamma_R^\theta$. The partial derivative with respect to λ_θ is

$$\frac{\partial \gamma_P^\theta}{\partial \lambda_\theta} = -\mu_2\eta \frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta} + \tilde{\mu}\frac{\partial \gamma_R^\theta}{\partial \lambda_\theta}.$$

Substitute $\frac{\partial \gamma_R^\theta}{\partial \lambda_\theta} = -(1+A_0)\eta \frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta}$ derived above

$$\frac{\partial \gamma_P^\theta}{\partial \lambda_\theta} = -\mu_2\eta \frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta} - \tilde{\mu}(1+A_0)\eta \frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta}.$$

Factoring out $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta}$,

$$\frac{\partial \gamma_P^\theta}{\partial \lambda_\theta} = -\left[\mu_2\eta + \tilde{\mu}(1+A_0)\eta\right] \frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta}.$$

Substitute back $\tilde{\mu} = \mu_1 + \mu_2 \frac{\beta_2}{\beta_1+\beta_2}$:

$$\frac{\partial \gamma_P^\theta}{\partial \lambda_\theta} = -\left[\mu_2\eta + \left(\mu_1 + \mu_2 \frac{\beta_2}{\beta_1+\beta_2}\right)(1+A_0)\eta\right] \frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta}$$

Let $K_P \equiv \mu_2\eta + \left(\mu_1 + \mu_2 \frac{\beta_2}{\beta_1+\beta_2}\right)(1+A_0)\eta$. As $\mu_1, \mu_2, \eta > 0$ and $(1+A_0) > 0$, then $K_P > 0$. Thus, the relationship is:

$$\frac{\partial \gamma_P^\theta}{\partial \lambda_\theta} = -K_P \frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta}$$

This result indicates that the derivative of the pollution growth rate γ_P^θ with respect to λ_θ has the opposite sign to the derivative of the labor share in final goods production $L_F^{\theta,*}$ with respect to λ_θ.

Impact of Moving from Time Consistency to Time-Inconsistency on γ_P^θ. We now combine the sensitivity analysis with Proposition 1, which describes how λ_θ changes when moving from time consistency (TC), where $\lambda_\theta(\sigma = 1) = \hat{\rho}$, to time-inconsistency (TI), where $\lambda_\theta(\sigma \neq 1) \neq \hat{\rho}$. Let $\gamma_P^\theta(\lambda)$ denote the steady-state pollution growth rate as a function of λ_θ. The TC rate is $\gamma_P^{*,TC} = \gamma_P^\theta(\hat{\rho})$ and the TI rate is $\gamma_P^{*,TI} = \gamma_P^\theta(\lambda_\theta^{ces})$. The change in the pollution growth rate, $\Delta\gamma_P^\theta = \gamma_P^{*,TI} - \gamma_P^{*,TC}$, can be seen using a first-order Taylor expansion:

$$\Delta\gamma_P^\theta \approx \left.\frac{\partial \gamma_P^\theta}{\partial \lambda_\theta}\right|_{\lambda_\theta=\hat{\rho}} (\lambda_\theta^{ces} - \hat{\rho}).$$

For simplicity in the exposition below, we denote $\frac{\partial \gamma_P^\theta}{\partial \lambda_\theta}\big|_{\lambda_\theta=\hat{\rho}}$ as $\frac{\partial \gamma_P^\theta}{\partial \lambda_\theta}$, assuming its sign is determined consistently.

Case $\sigma > 1$

According to Proposition 1, if $\sigma > 1$, then $\lambda_\theta^{ces} < \hat{\rho}$. Thus, the change $\Delta\lambda_\theta = \lambda_\theta^{ces} - \hat{\rho} < 0$. From Section A.5.2, for $\sigma > 1$, $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta} > 0$. From Section A.5.5, since $K_P > 0$ and $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta} > 0$, it follows that $\frac{\partial \gamma_P^\theta}{\partial \lambda_\theta} = -K_P \times (\text{positive quantity}) < 0$. Therefore, the change in pollution growth is

$\Delta\gamma_P^\theta \approx$ (negative derivative) $\times$ (negative change in λ_θ) > 0. Conclusion for $\sigma > 1$: Moving to time-inconsistency leads to an increase in the steady-state growth rate of pollution ($\gamma_P^{*,TI} > \gamma_P^{*,TC}$).

Case $\sigma < 1$

According to Proposition 1, if $\sigma < 1$, then $\lambda_\theta^{ces} > \hat{\rho}$. Thus, the change $\Delta\lambda_\theta = \lambda_\theta^{ces} - \hat{\rho} > 0$. From Section A.5.2, for $\sigma < 1$, the sign of $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta}$ depends on $C_1 = \left(\frac{1-\sigma}{\sigma}\right) H_2 \mathcal{K}_\lambda$.

- **Subcase 2a:** $C_1 > 1$
  - $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta} < 0$ (from Section A.5.2).
  - $\frac{\partial \gamma_P^\theta}{\partial \lambda_\theta} = -K_P \times$ (negative quantity) > 0 (from Section A.5.5).
  - $\Delta\gamma_P^\theta \approx$ (positive derivative) $\times$ (positive change in λ_θ) > 0.
  - Conclusion for $\sigma < 1, C_1 > 1$: $\gamma_P^{*,TI} > \gamma_P^{*,TC}$. Pollution growth increases, and time-inconsistency leads to higher pollution growth.
- **Subcase 2b:** $C_1 < 1$
  - $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta} > 0$ (from Section A.5.2).
  - $\frac{\partial \gamma_P^\theta}{\partial \lambda_\theta} = -K_P \times$ (positive quantity) < 0 (from Section A.5.5).
  - $\Delta\gamma_P^\theta \approx$ (negative derivative) $\times$ (positive change in λ_θ) < 0.
  - Conclusion for $\sigma < 1, C_1 < 1$: $\gamma_P^{*,TI} < \gamma_P^{*,TC}$. Pollution growth decreases, meaning that time-inconsistency leads to lower pollution growth.
- **Subcase 2c:** $C_1 = 1$
  - $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta} = 0$ (from Section A.5.2).
  - $\frac{\partial \gamma_P^\theta}{\partial \lambda_\theta} = 0$ (from Section A.5.5).
  - $\Delta\gamma_P^\theta \approx 0$.
  - Conclusion for $\sigma < 1, C_1 = 1$: $\gamma_P^{*,TI} \approx \gamma_P^{*,TC}$. The growth rate of pollution is not significantly affected by the time-inconsistency of the households.

Case $\sigma = 1$ (Logarithmic Utility) From Section A.5.2, for $\sigma = 1$, $\frac{\partial L_F^{\theta,*}}{\partial \lambda_\theta} = \frac{1}{(1+A_0)\eta} > 0$ (assuming $(1 + A_0)\eta \neq 0$). As stated in Section A.5.2 (and from Proposition 1), for $\sigma = 1$, $\lambda_\theta^{ces} = \hat{\rho}$. This means $\Delta\lambda_\theta = \lambda_\theta^{ces} - \hat{\rho} = 0$. Therefore, since λ_θ does not change, the pollution growth rate γ_P^θ also does not change:

$$\Delta\gamma_P^\theta = \gamma_P^\theta(\lambda_\theta^{ces}) - \gamma_P^\theta(\hat{\rho}) = \gamma_P^\theta(\hat{\rho}) - \gamma_P^\theta(\hat{\rho}) = 0.$$

Therefor, for $\sigma = 1$, one gets that $\gamma_P^{*,TI} = \gamma_P^{*,TC}$. Time-inconsistency has no impact on the pollution growth rate.

The derivations above show the claims made in Theorem 1. The impact of time-inconsistency on steady-state pollution growth critically depends on the elasticity of intertemporal substitution σ. For $\sigma > 1$, time-inconsistency unambiguously leads to higher growth rates, more labor allocated to the R&D sector, a higher growth rate of new ideas, a less agressive extraction of the resource, and a higher growth rate of pollution. For $\sigma < 1$, the outcome is conditional on the relative magnitudes of underlying model parameters, encapsulated in the term C_1. For $\sigma = 1$, there is no effect, and it is completely irrelevant if households show time-inconsistency or not. These findings highlight an interesting channel through which deviations from time-consistent preferences can affect economic growth, creation of ideas, extraction of the resource, the allocation of labor to the R&D sector, and the dynamics of pollution.

To illustrate the impact of moving from time-consistent to time-inconsistent households, we summarize all of the key steady-state results presented until now in Table A.1.

Table A.1: Impact of a Shift from Time Consistency (TC) to Time-Inconsistency (TI) on Key Steady-State Variables

Variable	**Change in Variable when Moving from TC to TI ($X^{TI} - X^{TC}$)**				
	$\sigma > 1$	$\sigma < 1$			$\sigma = 1$
	($\lambda_\theta^{ces} < \hat{\rho}$)	($\lambda_\theta^{ces} > \hat{\rho}$)			($\lambda_\theta^{ces} = \hat{\rho}$)
		$C_1 < 1$	$C_1 > 1$	$C_1 = 1$	
Economic Growth Rate (γ^θ)	↑	↓	↓	↓	No Change
Labor in R&D Sector ($1 - L_F^{\theta,*}$)	↑	↓	↑	No Change	No Change
Resource Extraction Growth ($\gamma_R^\theta < 0$)	↑ (depletion rate $\lvert\gamma_R^\theta\rvert$ ↓)	↓ (depletion rate $\lvert\gamma_R^\theta\rvert$ ↑)	↑ (depletion rate $\lvert\gamma_R^\theta\rvert$ ↓)	No Change	No Change
Growth Rate of New Ideas (γ_N^θ)	↑	↓	↑	No Change	No Change
Pollution Growth Rate (γ_P^θ)	↑	↓	↑	No Change	No Change

Note: This table summarizes the first-order effects of a shift from time-consistent (TC) preferences (effective discount parameter $\hat{\rho}$) to time-inconsistent (TI) preferences (effective discount parameter λ_θ^{ces}). The relationship between λ_θ^{ces} and $\hat{\rho}$ depends on σ (elasticity of intertemporal substitution), as shown in Proposition 1. The condition $C_1 \equiv \left(\frac{1-\sigma}{\sigma}\right) H_2 \mathcal{K}_\lambda$ (where $A_0 = \frac{(1-\beta_1-\beta_2)(\beta_1+\beta_2)}{\beta_1}$, $\mathcal{K}_\lambda = \beta_1 + \beta_2 + \beta_2(1 + A_0) > 0$, and $H_2 > 0$) governs outcomes when $\sigma < 1$. Symbols: ↑ (increase in the variable), ↓ (decrease in the variable). For $\gamma_R^\theta < 0$: an increase (↑) means γ_R^θ becomes less negative, implying a decrease (↓) in the depletion rate $\lvert\gamma_R^\theta\rvert$; a decrease (↓) means γ_R^θ becomes more negative, implying an increase (↑) in the depletion rate $\lvert\gamma_R^\theta\rvert$. Derivations are based on Appendix A.5.

We have thus proved Theorem 1. □

B Additional Proofs and Derivations

B.1 Problem of the Representative Firm of the Resource Sector

The Hamiltonian of problem (1), with the dynamics constraint (2) is given by,

$$H^{nr} = \exp\left(-\int_t^s r(h)dh\right) P_R(s)R(s) + \lambda(s)\left[-R(s)\right], \tag{B.1}$$

with the correspondent F.O.C:

$$\frac{\partial H^{nr}}{\partial R(s)} = 0 \Leftrightarrow \exp\left(-\int_t^s r(h)dh\right) P_R(s) = \lambda(s),$$

$$\dot{\lambda}(s) = -\frac{\partial H^{nr}}{\partial S(s)} \Leftrightarrow \dot{\lambda}(s) = 0 \Leftrightarrow \lambda(s) = k_1,\ \forall k_1 \in \mathbb{R}.$$

Combining both previous equations gives $P_R(s) = k_1 \exp\left(\int_t^s r(h)dh\right)$. In order to get the growth rate of the prices, we firsts take natural logs and then differentiate with respect to time s, leading to

$$\gamma_{p_R} \equiv \frac{\dot{p_R}(s)}{p_R(s)} = r(s). \tag{B.2}$$

B.2 Problem of the Representative Firm of the Intermediate-Good Sector

In order to see the full self-contained problem of the representative firm of the intermediate-good sector, we write all the steps here.

Each intermediate good is only produced by a monopolist ν who charges a price $p^x(\nu, s)$. The design for such goods is invented in the R&D sector, developed in the paper. The present discounted value of owning the blueprint of a machine type ν is given by

$$V(\nu, s) = \int_s^\infty \exp\left(-\int_s^\tau r(\xi)d\xi\right) \pi(\nu, \tau)d\tau, \tag{B.3}$$

where $r(\xi)$ represents the monopolist's risk-free discount rate. The profit of monopolist ν at time s is defined as

$$\pi(\nu, s) := p^x(\nu, s)x(\nu, s) - rx(\nu, s). \tag{B.4}$$

In this case, we assume that one unit of machine ν can be produced at marginal cost $r > 0$ units of the final good (Scholz and Ziemes, 1999). Thus, any monopolist $\nu \in [0, N(s)]$ maximizes its profits defined in eq. (B.4). We also assume that the firm that discovers a new blueprint will receive a perpetual patent for this new invention, as it is normally assumed in the literature (Romer, 1990; Acemoglu, 2002).

Using the demand for a machine of type ν in eq. (9) and plugging it into the monopolist profit function gives the optimal price condition:

$$p^x(\nu,s) = \frac{r}{1-\beta_1-\beta_2} \equiv p^x \in \mathbb{R}_+ \text{ for all } \nu \text{ and } s. \tag{B.5}$$

Thus, any monopolist ν charges a constant rental price which is equal to a markup over the marginal cost r. Alternatively, one can write the first order condition of the monopolist as

$$r = (1-\beta_1-\beta_2)x(\nu,s)^{-\beta_1-\beta_2}L_F(s)^{\beta_1}R(s)^{\beta_2}. \tag{B.6}$$

Therefore, the profits of the monopolist producing input ν can be written as

$$\pi(\nu,s) = r\frac{(\beta_1+\beta_2)}{1-\beta_1-\beta_2}\cdot x(\nu,s) = (\beta_1+\beta_2)\cdot p^x \cdot x(\nu,s), \tag{B.7}$$

which is the same for all intermediate producers. Alternatively, the profits of intermediate firm $\nu \in [0N(s)]$ can also be expressed as

$$\pi(\nu,s) = \underbrace{(\beta_1+\beta_2)\left(\frac{r}{1-\beta_1-\beta_2}\right)^{-\frac{1-\beta_1-\beta_2}{\beta_1+\beta_2}}}_{\equiv B_0(r,\beta_1,\beta_2)=B_0\in\mathbb{R}_+} L_F(s)^{\beta_1/(\beta_1+\beta_2)}R(s)^{\beta_2/(\beta_1+\beta_2)}. \tag{B.8}$$

Before entering the intermediate-good market, a potential monopolist of variety ν must obtain a patent that will only allow her to produce the corresponding intermediate good. The entry decision will take the form of a two-step decision process. First, the potential monopolist will decide whether to enter the intermediate-good sector, and second, if she decides to enter, she will choose the optimal price for her blueprint ν. Thus, monopolist ν buys a patent at the initial time $t_0 < s$ from a firm in the R&D sector of infinite duration at a price of p_{RD} that allows her to produce the intermediate-good ν. The firm finances this acquisition by issuing bonds in the capital market for a price p_{RD}, which yields an interest $r(s)$ at each time s. Consequently, in order to produce its unique machinery, the firm must cover the entry cost p_{RD}. Thus, the threshold for entry into the intermediate-good market for monopolist ν is given by the condition that the price she will pay for a patent is equal to the discounted flow of profits:

$$\underbrace{\int_{t_0}^{\infty}\pi(\nu,s)e^{-\int_{t_0}^{s}r(\xi)d\xi}ds}_{V^m(\nu,s)=\text{Value Function of the Monopolist }\nu\text{ at initial time }t_0} = \underbrace{p_{RD}(t_0).}_{\text{Entry cost / Patent price}}$$

Alternatively, the previous expression can be expressed as

$$r(t_0)\ p_{RD}(t_0) - \frac{d}{dt_0}\left[p_{RD}(t_0)\right] = \pi(\nu,t_0),$$

which leads to the free-entry condition

$$r(t_0) = \gamma_{p_{RD}} + \frac{\pi(\nu, t_0)}{p_{RD}(t_0)}. \tag{B.9}$$

Therefore, plugging the optimal profits of the monopolist ν (eq. B.8) into eq. (B.3), we get the net present value of profit at time s,

$$V^m(\nu, s) = B_0(R, \beta_1, \beta_2) \int_s^\infty e^{-\int_s^{t_1} r(\xi)d\xi} L_F(\tau)^{\beta_1/(\beta_1+\beta_2)} R(\tau)^{\beta_2/(\beta_1+\beta_2)}\, d\tau.$$

This framework establishes the entry threshold: a potential intermediate goods producer will enter the market only if the discounted flow of profits is at least as large as the patent price.

B.3 Derivation of the Optimal Consumption Rule (22)

The Hamiltonian for the general CES function ($\sigma \neq 1$) is given by:

$$\begin{aligned}\mathcal{H}(s,t) &= \theta(s-t)\left[\frac{c_t(s)^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}}\right] \\ &+\lambda_t(s)\left[r_t(s)a_t(s) + w_t(s) + p_{R_t}(s) * R_t(s) - c_t(s)\right].\end{aligned} \tag{B.10}$$

Making use of the maximum principle, and following the approach in Strulik (2015) and Cabo et al. (2015), the first order condition for a maximum and the co-state variable give:

$$\frac{\partial \mathcal{H}}{\partial c_t(s)} \stackrel{!}{=} 0 \iff c_t(s) = \left[\frac{\theta(s-t)}{\lambda_t(s)}\right]^\sigma, \tag{B.11}$$

$$\dot{\lambda}_t(s) \stackrel{!}{=} -\left(\frac{\partial \mathcal{H}}{\partial a_t(s)}\right) \iff \dot{\lambda}_t(s) = -r_t(s)\lambda_t(s), \tag{B.12}$$

with the transversality condition (TVC) $\lim_{s\to\infty} \lambda(s)a(s) = 0$. Solving the differential equation (B.12), we obtain

$$\lambda_t(s) = \underbrace{\lambda_t(t)}_{\equiv\lambda_t\in\mathbb{R}} \exp\left\{-\int_t^s r_t(\xi)d\xi\right\}. \tag{B.13}$$

Now, plugging eq. (B.13) into the FOC of consumption (B.11) gives,

$$c_t(s) = \left[\frac{\theta(s-t)}{\lambda_t}\right]^\sigma \exp\left[\sigma \int_t^s r_t(\xi)d\xi\right]. \tag{B.14}$$

Inserting equations (B.14) into the budget constraint, and solving the corresponding differential equation for all $s \in [t, T)$, we get:

$$
\begin{aligned}
a_t(T) - \underbrace{a_t(t)}_{\equiv a_t} e^{\int_t^T r_t(\xi)d\xi} &= \int_t^T \left[w_t(\tau) + p_R(\tau) * R(\tau)\right] e^{\int_\tau^T r_t(\xi)d\xi} d\tau \\
&- \frac{1}{(\lambda_t)^\sigma} \int_t^T \theta(\tau - t)^\sigma \ \exp\left\{\sigma \int_t^\tau r_t(\xi)d\xi + \int_\tau^T r_t(\xi)d\xi\right\} d\tau.
\end{aligned} \tag{B.15}
$$

Dividing both sides of the equation by $e^{\int_t^T r_t(\xi)d\xi}$, one gets:

$$
\begin{aligned}
\frac{a_t(T)}{e^{\int_t^T r_t(\xi)d\xi}} &= a_t(t) + \int_t^T \left[w_t(\tau) + p_R(\tau) * R(\tau)\right] \frac{e^{\int_\tau^T r_t(\xi)d\xi}}{e^{\int_t^T r_t(\xi)d\xi}} d\tau \\
&- \frac{1}{(\lambda_t)^\sigma} \int_t^T \theta(\tau - t)^\sigma \ \frac{\exp\left\{\sigma \int_t^\tau r_t(\xi)d\xi + \int_\tau^T r_t(\xi)d\xi\right\}}{\exp\left\{\int_t^T r_t(\xi)d\xi\right\}} d\tau.
\end{aligned} \tag{B.16}
$$

Taking the limit when time $T \to \infty$, using the TVC $\lim_{T\to\infty}\left[a(T)\lambda_t e^{-\int_t^T r_t(\xi)d\xi}\right] = 0$, and simplifying the exponential terms, one gets,

$$
\begin{aligned}
0 &= a_t + \int_t^\infty \left[w_t(\tau) + p_R(\tau) * R(\tau)\right] \cdot e^{-\int_t^\tau r_t(\xi)d\xi} d\tau \\
&- \frac{1}{(\lambda_t)^\sigma} \cdot \int_t^\infty \theta(\tau - t)^\sigma \cdot \exp\left\{-(1-\sigma)\int_t^\tau r_t(\xi)d\xi\right\} d\tau.
\end{aligned}
$$

Solving for λ_t leads to

$$
\lambda_t = \left[\frac{\int_t^\infty \theta(\tau - t)^\sigma \cdot \exp\left\{-(1-\sigma)\int_t^\tau r_t(\xi)d\xi\right\} d\tau}{a_t + \int_t^\infty \left[w_t(\tau) + p_R(\tau) * R(\tau)\right] \cdot e^{-\int_t^\tau r_t(\xi)d\xi} d\tau}\right]^{\frac{1}{\sigma}}. \tag{B.17}
$$

Plugging eq. (B.17) into the consumption expression (B.14), we get the following consumption expression,

$$
c_t(s) = \theta(s-t)^\sigma \cdot e^{\sigma \int_t^s r_t(\xi)d\xi} \cdot \left(\frac{a_t + \int_t^\infty \left[w_t(\tau) + p_R(\tau) * R(\tau)\right] \cdot \exp\left\{-\int_t^\tau r_t(\xi)d\xi\right\} d\tau}{\int_t^\infty \theta(\tau - t)^\sigma \cdot \exp\left\{-(1-\sigma)\int_t^\tau r_t(\xi)d\xi\right\} d\tau}\right). \tag{B.18}
$$

As we are interested in the study of the naive agent, this means $s = t$ which helps with $\theta(s-t)^\sigma = \theta(0)^\sigma = 1^\sigma = 1$ to get,[39]

$$
c_t(t) = \left(\frac{a_t + \int_t^\infty \left[w_t(\tau) + p_R(\tau) * R(\tau)\right] \cdot \exp\left\{-\int_t^\tau r_t(\xi)d\xi\right\} d\tau}{\int_t^\infty \theta(\tau - t)^\sigma \cdot \exp\left\{-(1-\sigma)\int_t^\tau r_t(\xi)d\xi\right\} d\tau}\right). \tag{B.19}
$$

[39] When the naive agent makes a plan at the beginning t, she does not know that she will "procrastinate" or not follow the plan she had in mind. Thus, when thinking about the optimal path at time t, she sets the plan, but later on, at $s > t$ she will recalculate, and this new time will again be the initial time of the planning horizon. This is why we set $t = s$.

Observe that in the numerator, one could make use of the change of variable $\tau = s$, as time s does not appear and will be the decisions made in the future. This result is similar to the one obtained in Cabo et al. (2015), but we now consider a broader income that also comes from the resource sector,

$$c_t(t) = \left(\frac{a_t + \int_t^\infty \left[w_t(s) + p_R(s) * R(s) \right] \cdot \exp\left\{ - \int_t^s r_t(\xi) d\xi \right\} ds}{\int_t^\infty \theta(s-t)^\sigma \cdot \exp\left\{ -(1-\sigma) \int_t^s r_t(\xi) d\xi \right\} ds} \right). \tag{B.20}$$

Taking natural logs on both sides, differentiating with respect to initial time t (when the decision is made), and making use of the Leibniz rule, one gets the following modified Ramsey rule for naive agents:

$$\frac{\dot{c}(t)}{c(t)} = \sigma \left[r_t - \frac{\int_t^\infty \theta(s-t)^\sigma \overbrace{\left[-\frac{\frac{d\theta(s-t)}{dt}}{\theta(s-t)} \right]}^{\equiv \rho(j)} \cdot \exp\left\{ -(1-\sigma) \int_t^s r_t(\xi) d\xi \right\} ds}{\int_t^\infty \theta(s-t)^\sigma \exp\left\{ -(1-\sigma) \int_t^s r_t(\xi) d\xi \right\} ds} \right]. \tag{B.21}$$

Noting that the time distance between when the action takes place (s), and when the decision is made (t),we define $j \equiv s - t$, and using the instantaneous discount rate $\rho(j) \equiv -\dot{\theta}(j)/\theta(j)$, one could write the previous expression as,

$$\gamma_c^{ces} \equiv \frac{\dot{c}(t)}{c(t)} = \sigma \left[r_t - \lambda_\theta^{ces} \right], \tag{B.22}$$

$$\lambda_\theta^{ces} = \frac{\int_0^\infty \rho(j) \theta(j)^\sigma \exp\left\{ -(1-\sigma) \int_t^s r_t(\xi) d\xi \right\} dj}{\int_0^\infty \theta(i)^\sigma \exp\left\{ -(1-\sigma) \int_t^s r_t(\xi) d\xi \right\} di}. \tag{B.23}$$

B.4 Proof of Remark 1

See also this result for the neoclassical growth model in Barro (1999) and Cabo et al. (2015).

Proof. Starting with the definition of λ_θ^{log},

$$
\begin{aligned}
\lambda_{\theta}^{log} &= \int_0^{\infty} \rho(j) \frac{\theta(j)}{\int_0^{\infty} \theta(i)di}\, dj, \quad \text{as } \int_0^{\infty} \theta(i)di \text{ is not a function of } j, \\
&= \frac{1}{\int_0^{\infty} \theta(i)di} \int_0^{\infty} \rho(j)\theta(j)\, dj, \quad \text{as } \rho(j) = \left[-\frac{\dot{\theta}(j)}{\theta(j)}\right], \\
&= \frac{1}{\int_0^{\infty} \theta(i)di} \int_0^{\infty} \left[-\frac{\dot{\theta}(j)}{\cancel{\theta(j)}}\right] \cancel{\theta(j)}\, dj, \\
&= -\frac{1}{\int_0^{\infty} \theta(i)di} \int_0^{\infty} \dot{\theta}(j)\, dj, \quad \text{and from the Fundamental Theorem of Calculus,} \\
&= -\frac{1}{\int_0^{\infty} \theta(i)di} \left[\lim_{b\to\infty} \theta(b) - \theta(0)\right] = -\frac{1}{\int_0^{\infty} \theta(i)di} [0-1] \\
&= \frac{1}{\int_0^{\infty} \theta(i)di}.
\end{aligned}
$$

Making use of the equivalent present value (21) where $\hat{\rho} = \left[\int_0^{\infty} \theta(j)dj\right]^{-1}$ one gets

$$\lambda_{\theta}^{log} = \hat{\rho}.$$

This finding -that under log utility, a discount function capturing procrastination yields an *observational equivalence* with the standard exponential discounting model- was also found by Barro (1999). □

B.5 Proof of Corollary 1

Proof. Starting with the definition of λ_{θ}^{ces}, under the exponential discount case, $\theta(j) = e^{-\hat{\rho}j}$, one gets

$$\lambda_{Exp}^{ces} = \frac{\int_0^{\infty} \rho(j) \left[e^{-\hat{\rho}j}\right]^{\sigma} \exp\left\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\right\} dj}{\int_0^{\infty} \left[e^{-\hat{\rho}i}\right]^{\sigma} \exp\left\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\right\} di}.$$

Using the fact that $\rho(j) = -\frac{\dot{\theta}(j)}{\theta(j)} = \hat{\rho}$, under the exponential discount case, we can write

$$\lambda_{Exp}^{ces} = \hat{\rho}\frac{\int_0^{\infty} \left[e^{-\hat{\rho}j}\right]^{\sigma} \exp\left\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\right\} dj}{\int_0^{\infty} \left[e^{-\hat{\rho}i}\right]^{\sigma} \exp\left\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\right\} di} = \hat{\rho}.$$

This shows that under the exponential discount case, the effective rate of time preference λ_{Exp}^{ces} is equal to the discount rate $\hat{\rho}$, for any value of σ. □

B.6 Proof of Proposition 1 for the General Case: non-constant $r(t)$

We now provide a proof for Proposition 1, extending the result to the transition paths of the economy where the interest rate $r(s)$ is not constant, i.e., a function of time. We show that the effective rate of

time preference, λ_θ^{ces}, is a strictly monotonically decreasing function of the elasticity of intertemporal substitution, σ. This global property implies the proposition as a special case.

We first establish the convergence conditions for the integrals defining λ_θ^{ces} in a preliminary lemma. A key insight is that this convergence depends not on the entire path of interest rates, but on their long-run behavior, justifying a constant convergence threshold $\bar{\sigma}$.

Lemma 3 (Convergence Condition for the General Case)**.** *Let the interest rate $r(s)$ be a continuous function of time s, and assume the economy has positive growth rates such that $r(s)$ converges to a long-run value $r_\infty = \lim_{s\to\infty} r(s)$ and $r_\infty > \rho_\infty$, which is the case along the balanced growth path. We maintain the assumptions from the main text: $\rho(j)$ is non-increasing with $\lim_{j\to\infty}\rho(j) = \rho_\infty > 0$.*

The effective rate of time preference $\lambda_\theta^{ces}(\sigma, r(s))$ is well-defined if and only if $0 \leq \sigma < \bar{\sigma}$, where the threshold $\bar{\sigma}$ is given by:

$$\bar{\sigma} = \frac{r_\infty}{r_\infty - \rho_\infty} \tag{B.24}$$

Note that $\bar{\sigma} > 1$ since $r_\infty > \rho_\infty > 0$ implies $r_\infty > r_\infty - \rho_\infty$.

Proof. The effective rate of time preference λ is defined by the ratio of two integrals, $N(\sigma)/I(\sigma)$, where

$$N(\sigma) = \int_0^\infty \rho(j)\,\theta(j)^\sigma \exp\left\{-(1-\sigma)\int_t^{t+j} r(\xi)d\xi\right\} dj,$$
$$I(\sigma) = \int_0^\infty \theta(j)^\sigma \exp\left\{-(1-\sigma)\int_t^{t+j} r(\xi)d\xi\right\} dj.$$

Since $\lim_{j\to\infty}\rho(j) = \rho_\infty$ is a positive finite constant, the convergence of both integrals is identical by the Limit Comparison Test. We thus focus on the convergence of $I(\sigma)$. Let the integrand be denoted $h(j;\sigma)$. The convergence of $I(\sigma)$, $\int_0^\infty h(j;\sigma)dj$ is determined by the asymptotic behavior of $h(j;\sigma)$ for large j.

We will use the integral root test. Note that the integral converges if $\lim_{j\to\infty}[h(j;\sigma)]^{1/j} < 1$ and diverges if this limit is greater than 1. First, we express $h(j;\sigma)$ in exponential form using $\theta(j) = \exp(-\int_0^j \rho(s)ds)$:

$$h(j;\sigma) = \exp\left(-\sigma\int_0^j \rho(s)ds - (1-\sigma)\int_t^{t+j} r(\xi)d\xi\right)$$

Taking the $1/j$-th root is equivalent to dividing the exponent by j:

$$[h(j;\sigma)]^{1/j} = \exp\left(-\frac{\sigma}{j}\int_0^j \rho(s)ds - \frac{1-\sigma}{j}\int_t^{t+j} r(\xi)d\xi\right)$$

To evaluate the limit as $j\to\infty$, we use the property that for any function $f(x)$ converging to a limit f_∞, the limit of its Cesàro mean is also f_∞. That is, $\lim_{j\to\infty}\frac{1}{j}\int_a^{a+j} f(x)dx = f_\infty$. Applying this to our case, we get

$$\lim_{j\to\infty}\frac{1}{j}\int_0^j \rho(s)ds = \rho_\infty \quad \text{and} \quad \lim_{j\to\infty}\frac{1}{j}\int_t^{t+j} r(\xi)d\xi = r_\infty$$

The starting point t does not affect the limit of the average, which captures the long-run behavior. Therefore, the limit of the root is:

$$\lim_{j\to\infty}[h(j;\sigma)]^{1/j} = \exp(-\sigma\rho_\infty - (1-\sigma)r_\infty)$$

For the integral to converge, this limit must be strictly less than 1, which requires the exponent to be strictly negative:

$$-\sigma\rho_\infty - (1-\sigma)r_\infty < 0 \quad \Longrightarrow \quad \sigma\rho_\infty + (1-\sigma)r_\infty > 0$$

Rearranging for σ:

$$\sigma\rho_\infty + r_\infty - \sigma r_\infty > 0 \quad \Longrightarrow \quad r_\infty > \sigma(r_\infty - \rho_\infty)$$

Since we assume $r_\infty > \rho_\infty$, the term $(r_\infty - \rho_\infty)$ is positive. Dividing by it preserves the inequality:

$$\sigma < \frac{r_\infty}{r_\infty - \rho_\infty}$$

Thus, the integrals converge for $\sigma < \bar{\sigma} = \frac{r_\infty}{r_\infty - \rho_\infty}$ and diverge for $\sigma > \bar{\sigma}$. □

Proposition 4 (General Case with Time-Varying Interest Rate). *Under CES utility and a non-constant discount function, for $\sigma < 1$ we have $\lambda_\theta^{ces} > \hat{\rho}$. Conversely, $\lambda_\theta^{ces} < \hat{\rho}$ if $\sigma \in (1, \bar{\sigma})$, where $\bar{\sigma} = \frac{r_\infty}{r_\infty - \rho_\infty} > 1$.*

Proof. The proposition's claims are a direct consequence of a stronger result: that $\lambda_\theta^{ces}(\sigma)$ is a strictly monotonically decreasing function of σ for all $\sigma \in [0, \bar{\sigma})$. Since we know that $\lambda_\theta^{ces}(1) = \hat{\rho}$, proving this monotonicity is sufficient.

First, note that the effective rate of time preference, $\lambda(\sigma)$, is a weighted average of the instantaneous discount rates, $\rho(j)$. Let the weighting function $h(j;\sigma,t)$ be the kernel of the integrals defining $\lambda(\sigma)$. This function depends on the time horizon j, the EIS σ, and the decision time t:

$$h(j;\sigma,t) = \theta(j)^\sigma \exp\left\{-(1-\sigma)R(j;t)\right\}, \quad \text{where } R(j;t) := \int_t^{t+j} r(\xi)d\xi$$

Since $h(j;\sigma,t) > 0$ for all $j \geq 0$, we can define a σ-dependent probability density function $p(j;\sigma,t)$ over the time horizon $j \in [0,\infty)$:

$$p(j;\sigma,t) = \frac{h(j;\sigma,t)}{\int_0^\infty h(k;\sigma,t)dk} = \frac{h(j;\sigma,t)}{I(\sigma,t)}$$

With this definition, $\lambda(\sigma,t)$ is precisely the expected value of $\rho(j)$ with respect to this probability measure:

$$\lambda(\sigma,t) = \int_0^\infty \rho(j)p(j;\sigma,t)dj = E_{\sigma,t}[\rho(j)]$$

The subscripts on the expectation operator $E_{\sigma,t}$ emphasize that the underlying probability distribution changes with σ and depends on the decision time t.

As a second step, to prove *monotonicity*, we analyze the *sign of the derivative* $\partial\lambda(\sigma)/\partial\sigma$. We will now show that this derivative can be expressed as a statistical covariance. We start from the definition of $\lambda(\sigma,t)$ and apply the quotient rule:

$$\frac{\partial\lambda(\sigma,t)}{\partial\sigma} = \frac{\partial}{\partial\sigma}\left(\frac{N(\sigma,t)}{I(\sigma,t)}\right) = \frac{\frac{\partial N(\sigma,t)}{\partial\sigma}\cdot I(\sigma,t) - N(\sigma,t)\cdot\frac{\partial I(\sigma,t)}{\partial\sigma}}{[I(\sigma,t)]^2}$$

Using the Leibniz rule to differentiate under the integral sign, we get:

$$\frac{\partial I(\sigma,t)}{\partial\sigma} = \int_0^\infty \frac{\partial h(j;\sigma,t)}{\partial\sigma}dj \quad \text{and} \quad \frac{\partial N(\sigma,t)}{\partial\sigma} = \int_0^\infty \rho(j)\frac{\partial h(j;\sigma,t)}{\partial\sigma}dj$$

A key step is to simplify $\frac{\partial h}{\partial\sigma}$ using the logarithmic derivative identity, $\frac{\partial h}{\partial\sigma} = h\cdot\frac{\partial\ln h}{\partial\sigma}$. Let us define

$$g(j;t) := \frac{\partial\ln h(j;\sigma,t)}{\partial\sigma}.$$

In order to get that, we take the log of $h(.)$:

$$\ln h(j;\sigma,t) = \sigma\ln\theta(j) - (1-\sigma)R(j;t),$$

which gives us

$$g(j;t) = \frac{\partial\ln h(j;\sigma,t)}{\partial\sigma} = \ln\theta(j) + R(j;t) \tag{B.25}$$

Note that $g(j;t)$ does not depend on σ. The derivatives of the integrals are thus:

$$\frac{\partial I(\sigma,t)}{\partial\sigma} = \int_0^\infty h(j;\sigma,t)g(j;t)dj \quad \text{and} \quad \frac{\partial N(\sigma,t)}{\partial\sigma} = \int_0^\infty \rho(j)h(j;\sigma,t)g(j;t)dj$$

Substituting these explicit forms back into the quotient rule:

$$\frac{\partial\lambda(\sigma,t)}{\partial\sigma} =$$

$$\frac{\left(\int_0^\infty \rho(j)h(j;\sigma,t)g(j;t)dj\right)\left(\int_0^\infty h(k;\sigma,t)dk\right) - \left(\int_0^\infty \rho(k)h(k;\sigma,t)dk\right)\left(\int_0^\infty h(j;\sigma,t)g(j;t)dj\right)}{[I(\sigma,t)]^2}$$

Dividing the numerator and denominator by $[I(\sigma,t)]^2$ allows us to rewrite this expression in terms of expectations:

$$\frac{\partial\lambda(\sigma,t)}{\partial\sigma} = \frac{\int_0^\infty \rho(j)g(j;t)h(j;\sigma,t)dj}{I(\sigma,t)} - \left(\frac{\int_0^\infty \rho(k)h(k;\sigma,t)dk}{I(\sigma,t)}\right)\left(\frac{\int_0^\infty g(j;t)h(j;\sigma,t)dj}{I(\sigma,t)}\right)$$

Using the definition of the expectation $E_{\sigma,t}[X] = \int_0^\infty X(j)p(j;\sigma,t)dj$, this simplifies to:[40]

$$\frac{\partial\lambda(\sigma,t)}{\partial\sigma} = E_{\sigma,t}[\rho(j)g(j;t)] - E_{\sigma,t}[\rho(j)]E_{\sigma,t}[g(j;t)]$$

[40] See https://en.wikipedia.org/wiki/Law_of_the_unconscious_statistician.

This final expression is, by definition, the covariance between $\rho(j)$ and $g(j;t)$:

$$\frac{\partial \lambda(\sigma,t)}{\partial \sigma} = \mathrm{Cov}_{\sigma,t}(\rho(j), g(j;t))$$

The sign of the derivative is therefore determined by the sign of this covariance between $\rho(j)$ and $g(j;t)$, which we analyze next.

We now analyze the *monotonicity* of $\rho(j)$ and $g(j;t)$ with respect to the time horizon j.

The function $\boldsymbol{\rho(j)}$: By the core assumption of non-constant, present-biased discounting, the instantaneous discount rate $\rho(j)$ is a non-increasing function of j (i.e., $\dot{\rho}(j) \leq 0$). Since we consider a departure from standard exponential discounting, $\rho(j)$ is not a constant function.

The function $\boldsymbol{g(j;t)}$: We examine its derivative of $g(j;t)$ with respect to j:

$$\frac{\partial g(j;t)}{\partial j} = \frac{\partial}{\partial j}(\ln \theta(j)) + \frac{\partial}{\partial j}\left(\int_t^{t+j} r(\xi) d\xi\right)$$

The first term is $\frac{\dot{\theta}(j)}{\theta(j)} = -\rho(j)$. The second term, by the Leibniz rule for differentiation under the integral sign, is $r(t+j)$. Thus:

$$\frac{\partial g(j;t)}{\partial j} = r(t+j) - \rho(j) \tag{B.26}$$

The term $r(t+j) - \rho(j)$ has a clear economic interpretation: it is the *net return to waiting*. In the classical Ramsey model, where the interest is constant and agents have exponential discounting, it is simply $r - \rho$. Here, with time-varying interest rates and non-exponential discounting, it becomes $r(t+j) - \rho(j)$. For a household to have an incentive to save and postpone consumption from time t to time $t+j$, the marginal benefit of saving, the real interest rate $r(t+j)$, must exceed the marginal cost of delaying gratification, which is captured by the instantaneous discount rate $\rho(j)$. If $r(t+j) < \rho(j)$, no rational agent would save for that period, as the reward for waiting would be insufficient to compensate for their impatience. A viable economy with positive savings and growth must therefore satisfy $r(s) > \rho(s-t)$ along its optimal path. This makes the assumption $r(t+j) > \rho(j)$ not merely a mathematical convenience but a necessary condition for positive growth. Consequently, $\frac{\partial g(j;t)}{\partial j} > 0$, meaning $g(j;t)$ is a *strictly increasing* function of j.

Finally, we have established that $\rho(j)$ is a non-increasing function and $g(j;t)$ is a strictly increasing function of j. A fundamental theorem of statistics (related to Chebyshev's Sum Inequality) states that the covariance between a non-increasing function and a non-decreasing function is non-positive. Because $g(j;t)$ is strictly increasing and $\rho(j)$ is not constant, the covariance is strictly negative:

$$\frac{\partial \lambda(\sigma)}{\partial \sigma} = \mathrm{Cov}_{\sigma}(\rho(j), g(j;t)) < 0.$$

Since the derivative of $\lambda(\sigma)$ with respect to σ is strictly negative for all σ in the convergence domain $[0, \bar{\sigma})$, the function $\lambda(\sigma)$ is *strictly monotonically decreasing* in σ.

This global monotonicity proves the proposition. Given that $\lambda(1) = \hat{\rho}$, then:

- For $\sigma < 1$, monotonicity implies $\lambda(\sigma) > \lambda(1)$, which means $\lambda_\theta^{ces} > \hat{\rho}$.
- For $\sigma \in (1, \bar{\sigma})$, monotonicity implies $\lambda(\sigma) < \lambda(1)$, which means $\lambda_\theta^{ces} < \hat{\rho}$.

This completes the proof for the general case of a time-varying interest rate, demonstrating that the proposition holds robustly along the economy's transition paths. □

B.7 Proof of Corollary 4. Log Markovian extraction

Proof. One just need to prove that $\gamma_R^{log} = -\hat{\rho}$. From Proposition the log case in 3 one gets the the growth rate of the extraction of the resource γ_R^{log}. Substituting the value of the labor force used in the final labor in the steady state L_F^{log} and simplifying leads to

$$\gamma_R^{log} = -\lambda^{log}.$$

Therefore, all that remains now is to prove that $\lambda^{log} = \hat{\rho}$, which has been proved in Remark 1. □

B.8 Proof of Corollary 5

Proof. Here we will show the case for $\sigma < 1$. The case for $\sigma > 1$ is obtained similarly.

$$\frac{R^{log}(s)}{S^{log}(s)} = \underbrace{-\underbrace{\gamma_R^{log}}_{<0}}_{>0} < \frac{R^{\theta}_{\sigma<1}(s)}{S^{\theta}_{\sigma<1}(s)} = \underbrace{-\underbrace{\gamma_R^{\theta,\sigma<1}}_{<0}}_{>0},$$

and using corollaries 3 and 4,

$$\hat{\rho} < \lambda_\theta^{\sigma<1} - \gamma^{\sigma<1} \cdot \left(\frac{\sigma-1}{\sigma}\right).$$

As for $\sigma < 1$, $\hat{\rho} < \lambda_\theta^{\sigma<1}$ (see Proposition 1),

$$\underbrace{\hat{\rho} - \lambda_\theta^{\sigma<1}}_{<0} < \underbrace{-\gamma^{\theta,\sigma<1} \cdot \underbrace{\left(\frac{\sigma-1}{\sigma}\right)}_{<0}}_{>0}.$$

which proofs that the LHS (negative) is smaller than the RHS (positive). □

B.9 Proof of Corollary 6

In this proof, one should interpret a general discount function $\theta(j)$ different than the exponential one. We show that this inequality holds for $\sigma < 1$, and is reversed for $\sigma > 1$. From

$$\frac{R_{Exp}^{\sigma<1}(s)}{S_{Exp}^{\sigma<1}(s)} < \frac{R_{\theta}^{\sigma<1}(s)}{S_{\theta}^{\sigma<1}(s)},$$

and using the Markovian strategy in Corollary 2, one can write

$$-\gamma_R^{Exp,\sigma<1} < -\gamma_R^{\theta,\sigma<1},$$

and thus,

$$\gamma_R^{\theta,\sigma<1} < \gamma_R^{Exp,\sigma<1}.$$

Using Corollary 3 involving the growth rate of the economy $\gamma^{\theta,\sigma<1}$, it leads to

$$\gamma^{\theta,\sigma<1} \cdot \left(\frac{\sigma-1}{\sigma}\right) - \lambda_{\theta}^{\sigma<1} < \gamma^{Exp,\sigma<1} \cdot \left(\frac{\sigma-1}{\sigma}\right) - \underbrace{\lambda_{Exp}^{\sigma<1}}_{=\hat{\rho}},$$

where $\lambda_{Exp} = \hat{\rho}$ for any σ by corollary 1. Thus,

$$\hat{\rho} - \lambda_{\theta}^{\sigma<1} < \left(\frac{\sigma-1}{\sigma}\right)\left[\gamma^{Exp,\sigma<1} - \gamma^{\theta,\sigma<1}\right].$$

Hence, keeping in mind we are under the $\sigma < 1$ case,

$$0 < \gamma_R^{Exp,\sigma<1} - \gamma_R^{\theta,\sigma<1} \iff \gamma_R^{Exp} - \gamma_R^{\theta} = \frac{\sigma-1}{\sigma}(\gamma^{Exp} - \gamma^{\theta}) - (\hat{\rho} - \lambda_{\theta}).$$

Using the growth rates of the economy in Proposition 3,

$$\gamma^{Exp} - \gamma^{\theta} = H_2\, B\, (\lambda_{\theta} - \hat{\rho}), \quad H_2 > 0,\ B = \beta_1 + \beta_2 + \beta_2(1 + A_0) > 0,$$

so

$$\gamma_R^{Exp} - \gamma_R^{\theta} = (\lambda_{\theta} - \hat{\rho})\left[1 + \tfrac{\sigma-1}{\sigma} H_2\, B\right].$$

We must show that the braket is positive. Define

$$D = \beta_1 + A_0(\sigma\beta_1 + \beta_2) + \beta_2(2 - \sigma), \quad H_2 = \frac{\sigma}{D}, \quad B = \beta_1 + 2\beta_2 + \beta_2 A_0.$$

Then

$$1 + \frac{\sigma-1}{\sigma} H_2 B = 1 + \frac{\sigma-1}{D} B = \frac{D + (\sigma-1)B}{D},$$

and

$$D + (\sigma-1)B = \sigma(1 + A_0)(\beta_1 + \beta_2) > 0,$$

so the bracket is > 0. Thus

$$\text{sign}(\gamma_R^{Exp} - \gamma_R^{\theta}) = \text{sign}(\lambda_\theta - \hat{\rho}).$$

Finally, Proposition 1 says

$$\sigma < 1 \implies \lambda_\theta > \hat{\rho}, \quad \sigma > 1 \implies \lambda_\theta < \hat{\rho}.$$

Hence, when $\sigma < 1$,

$$\gamma_R^{\text{Exp}} - \gamma_R^{\theta} > 0 \quad \implies \quad -\gamma_R^{\text{Exp}} < -\gamma_R^{\theta} \quad \implies \quad \frac{R_{\text{Exp}}}{S_{\text{Exp}}} < \frac{R_\theta}{S_\theta},$$

which shows that the extraction rate of an economy populated by the exponential discount agents (time-consisntent) is lower if the EIS is lower than one. If $\sigma > 1$, all inequalities reverse. This completes the proof. □

B.10 Derivation of the growth rate of Extraction over Resource left

We now derive eq. (B.33). From $\frac{R(s)}{S(s)}$, the time derivative of this quotient is given by

$$\left(\frac{\dot{R(s)}}{S(s)}\right) \equiv \frac{d}{ds}\left[\frac{R(s)}{S(s)}\right] = \frac{\dot{R}(s)S(s) - R(s)\dot{S}(s)}{S(s)^2} = \frac{\dot{R}(s)}{S(s)} - \frac{R(s)}{S(s)}\frac{\dot{S}(s)}{S(s)}.$$

From the dynamics of the resource, dividing both sides by $S(s)$ leads to $\frac{\dot{S}(s)}{S(s)} = -\frac{R(s)}{S(s)}$. Substituting it in the previous expressions gives

$$\frac{d}{ds}\left[\frac{R(s)}{S(s)}\right] = \frac{\dot{R(s)}}{S(s)} + \frac{R(s)}{S(s)}\frac{R(s)}{S(s)}.$$

Thus, the growth rate of $\frac{R(s)}{S(s)}$ is given by

$$\gamma_{\frac{R}{S}} \equiv \frac{\frac{d}{ds}\left[\frac{R(s)}{S(s)}\right]}{\frac{R(s)}{S(s)}} = \underbrace{\frac{\dot{R}(s)}{R(s)}}_{\equiv \gamma_R} + \frac{R(s)}{S(s)}. \tag{B.27}$$

Thus, the full system can be written as

$$\gamma_{\frac{c}{K}} = \frac{Y(s)}{K(s)}\left[\sigma(1-\beta_1-\beta_2)^2 - 1\right] - \sigma\lambda_\theta + \frac{c(s)}{K(s)}, \tag{B.28}$$

$$\gamma_{\frac{R}{S}} = \frac{R(s)}{S(s)} + \underbrace{(\beta_1+\beta_2)\frac{Y(s)}{K(s)} - \frac{c(s)}{K(s)} + \eta(1-L_F(s))\frac{\beta_2}{1-\beta_1-\beta_2} + \eta L_F(s)(\beta_1+\beta_2)}_{=\gamma_R}, \tag{B.29}$$

$$\gamma_{L_F} = (\beta_1+\beta_2)\frac{Y(s)}{K(s)} - \frac{c(s)}{K(s)} + \eta(1-L_F(s))\left[\frac{\beta_2 - (1-\beta_1-\beta_2)}{1-\beta_1-\beta_2}\right] + \eta L_F(s)\frac{(\beta_1+\beta_2)(1-\beta_2)}{\beta_1}, \tag{B.30}$$

$$\gamma_{\frac{Y}{K}} = (\beta_1+\beta_2)\eta L_F(s) - (1-\beta_1-\beta_2)(\beta_1+\beta_2)\frac{Y(s)}{K(s)} + \frac{\beta_2}{1-\beta_1-\beta_2}\eta\left(1-L_F(s)\right), \tag{B.31}$$

$$\gamma_P(s) = \mu_2\eta(1-L_F(s)) + \mu_2\frac{\beta_1}{\beta_1+\beta_2}\cdot\gamma_{L_F}(s) + \left(\mu_1+\mu_2\frac{\beta_2}{\beta_1+\beta_2}\right)\left(\gamma_{\frac{R}{S}}(s) - \frac{R(s)}{S(s)}\right). \tag{B.32}$$

B.11 Stability of the Steady State

In order to study the stability properties of the steady state, we use the system of differential equations that describe the BGP. From the dynamics of the non-renewable resource (1), one gets (see Online Appendix B.10).

$$\gamma_{\frac{R}{S}} = \gamma_R + \frac{R(s)}{S(s)}. \tag{B.33}$$

Working now with the system of differential equations that characterize the BGP and plugging in the previously derived equation (B.33) into the growth rate of the resource γ_R, one can solve the steady state of the system. For notational convenience, we first define

$$yk = \frac{Y}{K}, \qquad ck = \frac{C}{K}, \qquad rs = \frac{R}{S}.$$

We then linearize the system, obtain the Jacobian $\mathbf{J}_{4\times4}$, and examine its eigenvalues analytically. As we will show, the system is partially recursive, and the analysis of pollution can be left until the end. Throughout we adopt the compact notation

$$\beta \equiv \beta_1+\beta_2 \in (0,1), \qquad A_0 \equiv \frac{(1-\beta)\beta}{\beta_1} > 0, \qquad \gamma_{z_i} \equiv \dot{x}_i/x_i.$$

Rewriting then the system in Proposition 2, and using the definitions in the main text, we obtain the four autonomous differential equations (avoiding the time dependence of the variables to simplify notation)

$$\gamma_{ck} = \left[\sigma(1-\beta)^2 - 1\right]yk - \sigma\lambda_\theta + ck, \tag{B.34a}$$

$$\gamma_{yk} = \beta\eta L_F - \beta(1-\beta)yk + \frac{\beta_2}{1-\beta}\eta(1-L_F), \tag{B.34b}$$

$$\gamma_{L_F} = \beta yk - ck + \eta(1-L_F)\left[\frac{\beta_2-(1-\beta)}{1-\beta}\right] + \eta L_F\frac{\beta(1-\beta_2)}{\beta_1}, \tag{B.34c}$$

$$\gamma_{rs} = \beta yk - ck + \eta\left[\tfrac{\beta_2}{1-\beta}(1-L_F) + \beta L_F\right] + rs, \tag{B.34d}$$

and the pollution equation

$$\gamma_P = \mu_2\eta(1 - L_F) + \mu_2\frac{\beta_1}{\beta_1 + \beta_2}\,\gamma_{L_F} + \left(\mu_1 + \mu_2\tfrac{\beta_2}{\beta_1+\beta_2}\right)(\gamma_{rs} - rs). \tag{B.35}$$

Note that γ_P depends *only* on (yk, ck, L_F, rs), and not pollution itself. Thus, once the economic core is solved, the pollution path follows mechanically. Setting the four growth rates of the system to zero gives a linear system that can be solved sequentially (see (B.34)).

Lemma 4 (Balanced-growth steady state)**.** *Given H_2 and A_0 defined above, the steady state is unique and given by the common growth rate*

$$\gamma^* = \gamma^*(\beta_1, \beta_2, \eta, \sigma, \lambda_\theta) = H_2\Big\{\eta\beta A_0 - \lambda_\theta\big[\beta + \beta_2(1 + A_0)\big]\Big\} \tag{B.36}$$

and

$$L_F^* = L_F^*(\beta_1, \beta_2, \eta, \sigma, \lambda_\theta) = \frac{\eta + \lambda_\theta - \left(\frac{\sigma-1}{\sigma}\right)\gamma^*}{(1 + A_0)\eta}, \tag{B.37}$$

$$yk^* = yk^*(\beta_1, \beta_2, \eta, \sigma, \lambda_\theta) = \frac{\beta\eta L_F^* + \frac{\beta_2}{1-\beta}\eta(1 - L_F^*)}{\beta(1 - \beta)}, \tag{B.38}$$

$$ck^* = ck^*(\beta_1, \beta_2, \eta, \sigma, \lambda_\theta) = \sigma\lambda_\theta - yk^*\big[\sigma(1 - \beta)^2 - 1\big], \tag{B.39}$$

$$rs^* = rs^*(\beta_1, \beta_2, \eta, \sigma, \lambda_\theta) = -\gamma_R^* = (1 + A_0)\eta\, L_F^* - \eta. \tag{B.40}$$

Proof Sketch. Solve (B.34b)=0 for yk as a function of L_F, plug the result into (B.34a)=0 to obtain ck, insert (yk, ck) in (B.34c)=0 and solve for L_F; finally substitute L_F into (B.34d) $= 0$ to get rs. Algebraic manipulations yield the compact formulas above. As will be shown later, in equilibrium, the resource extraction ratio rs^* is constant and positive. We have decided to write the steady states also in a function form, to clearly show that they depend on the deep parameters of the model, and importantly, are sensitive to the time-inconsistency factor captured by λ_θ. □

When studying pollution, we have mentioned three possible scenarios: (i) pollution falls over time, (ii) pollution remains flat, and (iii) pollution continues to rise. The first two cases are the most interesting ones, as they are the ones that can be achieved by humankind. The third case is the one that we want to avoid. Using $\gamma_{L_F}^* = 0$ and (B.40) in (B.35),

$$\gamma_P^* = \gamma_P^*(\beta_1, \beta_2, \eta, \sigma, \lambda_\theta, \mu_1, \mu_2) = \mu_2\eta(1 - L_F^*) - \left(\mu_1 + \mu_2\tfrac{\beta_2}{\beta_1+\beta_2}\right) rs^*. \tag{B.41}$$

Note that several forces influence the growth rate of pollution. The first term, with $1 - L_F^*$ is the share of labor in R&D, and $\eta(1 - L_F^*)$ is the growth rate of new ideas/varieties. Thus, the first term links pollution generation (μ_2) to this "ideas" sector. The second term links pollution to the

extraction rate rs^*, or equivalently, to the negative growth rate of resource extraction γ_R^*. The term $\left(\mu_1 + \mu_2 \frac{\beta_2}{\beta_1+\beta_2}\right)$ is the combined pollution intensity associated with resource extraction, where μ_1 is the direct pollution intensity from extraction and $\mu_2 \frac{\beta_2}{\beta_1+\beta_2}$ is the indirect pollution intensity from the fact that the extracted resource (R) is used in final production (Y) with intensity μ_2, and $\frac{\beta_2}{\beta_1+\beta_2}$ is a weight related to the resource's importance in production relative to labor.

Before linearizing the system, let us define $\mathbf{z} = (yk, ck, L_F, rs)^\top$, and let $\mathbf{z}^* = (yk^*, ck^*, L_F^*, rs^*)^\top$ be the vector in the steady state, which we define with an asterisk. For any component of that vector z_i (where $z_1 = yk$, $z_2 = ck$, $z_3 = L_F$, $z_4 = rs$), its evolution over time s is given by $\dot{z}_i(s) = z_i(s) \times \gamma_{z_i}(\mathbf{z}(s))$. To study the local stability of the steady state $\mathbf{z}^*$, we linearize the system of differential equations $\dot{z}_i = z_i \gamma_{z_i}(\mathbf{z})$ around $\mathbf{z}^*$. Let $\Delta z_i = z_i - z_i^*$ represent the deviation of z_i from its steady-state value. The time evolution of these deviations is given by $\Delta \dot{z}_i = \dot{z}_i - \dot{z}_i^*$. Since z_i^* is a constant steady state, $\dot{z}_i^* = 0$. Thus $\Delta \dot{z}_i = \dot{z}_i$.

The Jacobian matrix $\mathbf{J}_{4\times4}$, evaluated at the steady state $\mathbf{z}^*$ is derived in Appendix B.12. The stability of the steady state is determined by the eigenvalues of the Jacobian matrix, which are the roots m of the characteristic equation $\det(\mathbf{J}_{4\times4} - m\mathbf{I}_4) = 0$, where $\mathbf{I}_4$ is the 4×4 identity matrix. All entries in the fourth column are zero except for the bottom entry $(J_{44} - m) = (rs^* - m)$. This yields:

$$\det(\mathbf{J}_{4\times4} - m\mathbf{I}_4) = (rs^* - m) \cdot (-1)^{4+4} \det B(m),$$

where $B(m)$ is the 3×3 minor obtained by removing the fourth row and fourth column. The four eigenvalues of the economic system are $m_4 = rs^*$ and the three roots of the cubic equation $m^3 - p_1 m^2 - p_2 m - p_3 = 0$, where p_1, p_2, p_3 are derived in the Appendix B.12.

B.11.1 Stability Conditions for the Economic Core

For the steady state $\mathbf{z}^*$ to be locally asymptotically stable, all four eigenvalues of $\mathbf{J}_{4\times4}$ must have negative real parts. That is i) it must hold that $m_4 = rs^* < 0$, and ii) the other three eigenvalues, which are roots of the cubic $Q(m) = m^3 - p_1 m^2 - p_2 m - p_3 = 0$, must also have negative real parts. Let $d_1 = -p_1$, $d_2 = -p_2$, and $d_3 = -p_3$. The cubic becomes $m^3 + d_1 m^2 + d_2 m + d_3 = 0$. According to the Routh-Hurwitz stability criterion for a cubic polynomial, all its roots have negative real parts if and only if:

$$d_1 > 0, \quad d_3 > 0, \quad \text{and} \quad d_1 d_2 > d_3. \tag{B.42}$$

Note that the condition $d_2 > 0$ is implied if $d_1 > 0, d_3 > 0$, and $d_1 d_2 - d_3 > 0$. Therefore, the balanced-growth steady state $\mathbf{z}^*$ is locally asymptotically stable if and only if $rs^* < 0$ and the coefficients d_1, d_2, d_3 (derived from p_1, p_2, p_3) satisfy the Routh-Hurwitz conditions (B.42).

We now discuss the implications of $rs^* \equiv R/S$, which is positive in the steady state ($rs^* > 0$), as the extraction rate cannot be negative. In this case, the eigenvalue $m_4 = rs^*$ is a positive real

eigenvalue. This immediately implies that the steady state $\mathbf{z}^*$ is not asymptotically stable. Instead, it is a saddle point. With $m_4 = rs^* > 0$, the system has at least one unstable direction. If the other three roots (from the cubic polynomial $Q(m) = 0$) have negative real parts (i.e., $d_1 > 0, d_3 > 0, d_1 d_2 > d_3$), then the steady state is a saddle point with one unstable dimension and a three-dimensional stable manifold.

B.11.2 Pollution Dynamics and Stability

The growth rate of pollution P is given by equation (B.35). At the steady state, we have $\gamma_{L_F}(\mathbf{z}^*) = 0$ and $\gamma_{rs}(\mathbf{z}^*) = 0$. Thus, the pollution growth rate at the steady state becomes a constant (see equation (B.41)). The differential equation for pollution is $\dot{P}(s) = P(s) \times \gamma_P(\mathbf{z}, \gamma_{L_F}(\mathbf{z}), \gamma_{rs}(\mathbf{z}))$. If we consider P as a fifth variable, the full system includes $\mathbf{z}$ and P. Since $\gamma_{yk}, \gamma_{ck}, \gamma_{L_F}, \gamma_{rs}$ do not depend on P, the 4×1 block of derivatives of $(z_i \gamma_{z_i})$ with respect to P in the full 5×5 Jacobian matrix is zero. Linearizing $\dot{P} = P \times \gamma_P$ around $(\mathbf{z}^*, P^*, \gamma^*_{L_F} = 0, \gamma^*_{rs} = 0)$:

$$\Delta\dot{P} = \frac{\partial(P \times \gamma_P)}{\partial P}\bigg|_* \Delta P + \sum_{j=1}^{4} \frac{\partial(P \times \gamma_P)}{\partial z_j}\bigg|_* \Delta z_j$$

The term $\frac{\partial(P\times\gamma_P)}{\partial P}\Big|_{\mathbf{z}^*,P^*} = \gamma_P(\mathbf{z}^*) + P^* \frac{\partial \gamma_P(\mathbf{z})}{\partial P}\Big|_{\mathbf{z}^*,P^*}$. Since γ_P does not explicitly depend on P itself, $\frac{\partial \gamma_P}{\partial P} = 0$. So, this derivative is simply γ^*_P. The full 5×5 Jacobian matrix $\mathbf{J}_{\text{full}}$ therefore has a block lower triangular structure (or block upper triangular if P is ordered first):

$$\mathbf{J}_{\text{full}} = \begin{pmatrix} \mathbf{J}_{4\times4} & \mathbf{0}_{4\times1} \\ \mathbf{J}_{P,\mathbf{z}} & \gamma^*_P \end{pmatrix}$$

where $\mathbf{J}_{P,\mathbf{z}}$ is a 1×4 row vector of partial derivatives $P^* \frac{\partial \gamma_P}{\partial z_j}\Big|_{\mathbf{z}^*}$. The eigenvalues of $\mathbf{J}_{\text{full}}$ are the eigenvalues of $\mathbf{J}_{4\times4}$ and the diagonal element γ^*_P. Thus, $m_5 = \gamma^*_P$ is the fifth eigenvalue of the system. The stability of the economic core (determined by $\mathbf{J}_{4\times4}$) is unaffected by the pollution dynamics. Pollution acts as a "slave" variable whose behavior is determined by the state of the economic core, determining then γ^*_P. As in the steady state L^*_F and rs^* do not depend on pollution, the growth rate of pollution in the steady state γ^*_P is thus constant. Consequently, $\dot{P}(s) = \gamma^*_P \times P(s)$, which gives the dynamics of pollution $P(s) = P_0 e^{\gamma^*_P \times s}$. Three distinct long-run scenarios for pollution emerge, where $\lim_{s\to\infty} P(s) \equiv P^\theta_\infty$ is the steady state limit: *i)* $\gamma^*_P < 0$ (**UN / Humankind goal**), where pollution declines exponentially and converges to zero, $P^\theta_\infty = 0$; *ii)* $\gamma^*_P = 0$ (**Business-as-Usual**), where pollution stays at its current level, $P^\theta_\infty = P^{\theta,*} = P_0$; *iii)* $\gamma^*_P > 0$ (**CHKT**), where pollution grows without bound, $P^\theta_\infty = +\infty$.

The long-run behavior of pollution is thus entirely determined by the sign of γ^*_P, which is determined by the steady-state values of the economic variables L^*_F and rs^*, which in turn depend on the

model's deep parameters. Summarizing, $P_{\infty}^{\theta} \in \{0,\ P^{\theta,*},\ \infty\}$, is fully determined by the sign of γ_P^*. As the pollution dynamics does not feed back into the economic core, the four eigenvalues that govern (yk, ck, L_F, rs) are unchanged, and γ_P^* simply adds a fifth eigenvalue. Thus, the full 5×5 Jacobian is lower triangular and adds $m_5 = \gamma_P^*$ as an *independent* eigenvalue.

Therefore, the stability of the system comes from the study of the eigenvalues of the Jacobian matrix $\mathbf{J}_{\text{full}}$. One eigenvalue is directly $rs^* > 0$, which is positive. Thus, the equilibrium is a saddle point already by the fact that this economy has natural resources. The other three eigenvalues are determined by a cubic equation whose coefficients p_1, p_2, p_3 (defined in the Appendix B.12) depend on the system parameters and steady-state values. The Routh-Hurwitz conditions (B.42) applied to these coefficients will determine the nature of these three eigenvalues. Pollution dynamics introduce an additional, independent eigenvalue $m_5 = \gamma_P^*$, which dictates the long-term behavior of pollution without altering the stability characteristics of the economic variables. Only if pollution decreases over time, it contributes with a stable subspace that converges towards the steady state. If pollution is increasing, it is a source of instability, as it is a positive eigenvalue. If pollution is constant ($\gamma_P^* = 0$), the eigenvalue is zero, indicating that pollution levels persist without diverging or converging. Importantly, if agents are time-inconsistent, it may push the edge case of going from a negative eigenvalue to a positive one.

One can then get a threshold value for $\gamma_P^*(\beta_1, \beta_2, \eta, \sigma, \lambda_\theta, \mu_1, \mu_2)$ whose sign determines the dimension of the unstable manifold (but can never lead to a locally stable equilibrium, as there is already a positive eigenvalue $rs^* > 0$). In particular, pollution will decline (and thus the corresponding eigenvalue is negative, which implies that its eigendirection lies in the stable manifold) if and only if

$$\bar{\mu}_1 \equiv \mu_2 \left[\frac{\eta \left(1 - L_F^*(\beta_1, \beta_2, \eta, \sigma, \lambda_\theta)\right)}{rs^*(\beta_1, \beta_2, \eta, \sigma, \lambda_\theta)} - \frac{\beta_2}{\beta_1 + \beta_2} \right] < \mu_1,$$

where $\bar{\mu}_1$ is the threshold above which pollution converges back to the steady state (see Figure B.2).[41]

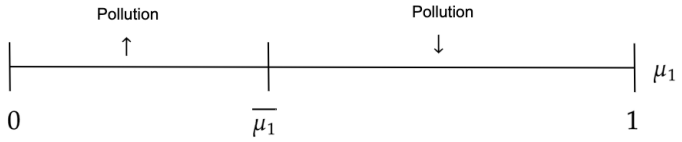


Figure B.2: Threshold μ_1 determines if pollution increases or decreases.

Assuming that humankind succeeds in achieving its pollution targets, set by various international agreements, one would expect that all pollution derived from economic activity would decrease and that only the so-called "natural pollution" would exist. One might consider pre-industrial pollution, for instance, generated by volcanic eruptions or by the environment itself. This would be the effect of

[41] Alternatively, the pollution parameters ratio $\frac{\mu_1}{\mu_2}$ should be greater than a given threshold, that is $\left[\frac{\eta(1-L_F^*(.))}{rs^*(.)} - \frac{\beta_2}{\beta_1+\beta_2}\right] < \frac{\mu_1}{\mu_2}$.

decreasing pollution until it converges to its natural earth level or its pre-industrial level. Thus, the fact that pollution decreases towards zero in our model should be interpreted as converging to its natural level or no extra pollution from human beings.

B.12 Jacobian, characteristic polynomial, and Routh–Hurwitz tests. Subsection B.11

Let $f_i(\mathbf{z}) = z_i\,\gamma_{z_i}(\mathbf{z})$, so that $\dot{z}_i = f_i(\mathbf{z})$. A first-order Taylor expansion of $f_i(\mathbf{z})$ around $\mathbf{z}^*$ yields

$$f_i(z) \approx f_i(z^*) + \sum_{j=1}^{4} \left.\frac{\partial f_i}{\partial z_j}\right|_{z^*} (z_j - z_j^*).$$

At steady state,

$$f_i(z^*) = z_i^*\,\gamma_{z_i}(z^*) = z_i^* \cdot 0 = 0.$$

Therefore, the linearized system is

$$\Delta\dot{z}_i = \sum_{j=1}^{4} \left.\frac{\partial f_i}{\partial z_j}\right|_{z^*} \Delta z_j.$$

The partial derivatives form the elements J_{ij} of the Jacobian matrix $\mathbf{J}$ are

$$J_{ij} = \left.\frac{\partial f_i}{\partial z_j}\right|_{z^*} = \left.\frac{\partial\big(z_i\,\gamma_{z_i}(z)\big)}{\partial z_j}\right|_{z^*}.$$

Using the product rule and evaluating at z^*, where $\gamma_{z_i}(z^*) = 0$, gives[42]

$$J_{ij} = \delta_{ij}\,\gamma_{z_i}(z^*) + z_i^* \left.\frac{\partial \gamma_{z_i}}{\partial z_j}\right|_{z^*} = z_i^* \left.\frac{\partial \gamma_{z_i}}{\partial z_j}\right|_{z^*}.$$

where δ_{ij} is the Kronecker delta. Thus the linearized system can be written compactly in matrix form as

$$\Delta\dot{\mathbf{z}} \;=\; \mathbf{J}(\mathbf{z}^*)\,\Delta\mathbf{z}.$$

B.12.1 Jacobian entry-by-entry

For notational convinience, we first define

$$\Xi_1 = \beta - \frac{\beta_2}{1-\beta} = \frac{\beta(1-\beta)-\beta_2}{1-\beta}, \tag{B.43}$$

$$\Xi_2 = \frac{\beta(1-\beta_2)}{\beta_1} - \frac{\beta_2-(1-\beta)}{1-\beta}. \tag{B.44}$$

We now derive each element J_{ij} of the 4×4 Jacobian matrix. Let $z_1 = yk, z_2 = ck, z_3 = L_F$, $z_4 = rs$.

[42] Using the product rule,

$$\frac{\partial\,(z_i\gamma_{z_i})}{\partial z_j} = \frac{\partial z_i}{\partial z_j}\,\gamma_{z_i} + z_i\,\frac{\partial \gamma_{z_i}}{\partial z_j} = \delta_{ij}\,\gamma_{z_i} + z_i\,\frac{\partial \gamma_{z_i}}{\partial z_j}.$$

Row 1: $f_1(\mathbf{z}) = yk \cdot \gamma_{yk}(\mathbf{z})$ The growth rate from (B.34b) is $\gamma_{yk} = \beta\eta L_F - \beta(1-\beta)yk + \frac{\beta_2}{1-\beta}\eta(1 - L_F)$. Rewriting for clarity on L_F: $\gamma_{yk} = \left(\beta\eta - \frac{\beta_2\eta}{1-\beta}\right) L_F - \beta(1-\beta)yk + \frac{\beta_2\eta}{1-\beta}$.

- $J_{11} = yk^* \frac{\partial \gamma_{yk}}{\partial yk}\Big|_{\mathbf{z}^*} = yk^*[-\beta(1-\beta)] = -\beta(1-\beta)yk^*$
- $J_{12} = yk^* \frac{\partial \gamma_{yk}}{\partial ck}\Big|_{\mathbf{z}^*} = yk^*[0] = 0$
- $J_{13} = yk^* \frac{\partial \gamma_{yk}}{\partial L_F}\Big|_{\mathbf{z}^*} = yk^*\left[\beta\eta - \frac{\beta_2\eta}{1-\beta}\right] = yk^*\eta\left(\beta - \frac{\beta_2}{1-\beta}\right)$. As $\Xi_1 \equiv \beta - \frac{\beta_2}{1-\beta} = \frac{\beta(1-\beta)-\beta_2}{1-\beta}$, then $J_{13} = \eta\Xi_1 yk^*$.
- $J_{14} = yk^* \frac{\partial \gamma_{yk}}{\partial rs}\Big|_{\mathbf{z}^*} = yk^*[0] = 0$

Row 2: $f_2(\mathbf{z}) = ck \cdot \gamma_{ck}(\mathbf{z})$ The growth rate from (B.34a) is $\gamma_{ck} = \left[\sigma(1-\beta)^2 - 1\right]yk - \sigma\lambda_\theta + ck$.

- $J_{21} = ck^* \frac{\partial \gamma_{ck}}{\partial yk}\Big|_{\mathbf{z}^*} = ck^*[\sigma(1-\beta)^2 - 1]$
- $J_{22} = ck^* \frac{\partial \gamma_{ck}}{\partial ck}\Big|_{\mathbf{z}^*} = ck^*[1] = ck^*$
- $J_{23} = ck^* \frac{\partial \gamma_{ck}}{\partial L_F}\Big|_{\mathbf{z}^*} = ck^*[0] = 0$
- $J_{24} = ck^* \frac{\partial \gamma_{ck}}{\partial rs}\Big|_{\mathbf{z}^*} = ck^*[0] = 0$

Row 3: $f_3(\mathbf{z}) = L_F \cdot \gamma_{L_F}(\mathbf{z})$ The growth rate from (B.34c) is $\gamma_{L_F} = \beta yk - ck + \eta(1 - L_F)\left[\frac{\beta_2-(1-\beta)}{1-\beta}\right] + \eta L_F \frac{\beta(1-\beta_2)}{\beta_1}$. Rewriting for clarity on L_F: $\gamma_{L_F} = \beta yk - ck + \eta\left[\frac{\beta_2-1+\beta}{1-\beta}\right] + \eta L_F\left(\frac{\beta(1-\beta_2)}{\beta_1} - \frac{\beta_2-1+\beta}{1-\beta}\right)$.

- $J_{31} = L_F^* \frac{\partial \gamma_{L_F}}{\partial yk}\Big|_{\mathbf{z}^*} = L_F^*[\beta] = \beta L_F^*$
- $J_{32} = L_F^* \frac{\partial \gamma_{L_F}}{\partial ck}\Big|_{\mathbf{z}^*} = L_F^*[-1] = -L_F^*$
- $J_{33} = L_F^* \frac{\partial \gamma_{L_F}}{\partial L_F}\Big|_{\mathbf{z}^*} = L_F^*\eta\left(\frac{\beta(1-\beta_2)}{\beta_1} - \frac{\beta_2-1+\beta}{1-\beta}\right)$. As $\Xi_2 \equiv \frac{\beta(1-\beta_2)}{\beta_1} - \frac{\beta_2-(1-\beta)}{1-\beta}$, thus, $J_{33} = \eta\Xi_2 L_F^*$.
- $J_{34} = L_F^* \frac{\partial \gamma_{L_F}}{\partial rs}\Big|_{\mathbf{z}^*} = L_F^*[0] = 0$

Row 4: $f_4(\mathbf{z}) = rs \cdot \gamma_{rs}(\mathbf{z})$ The growth rate from (B.34d) is $\gamma_{rs} = \beta yk - ck + \eta\Big[\frac{\beta_2}{1-\beta}(1 - L_F) + \beta L_F\Big] + rs$. Rewriting for clarity on L_F: $\gamma_{rs} = \beta yk - ck + \frac{\eta\beta_2}{1-\beta} + \eta\left(\beta - \frac{\beta_2}{1-\beta}\right) L_F + rs$.

- $J_{41} = rs^* \frac{\partial \gamma_{rs}}{\partial yk}\Big|_{\mathbf{z}^*} = rs^*[\beta] = \beta rs^*$
- $J_{42} = rs^* \frac{\partial \gamma_{rs}}{\partial ck}\Big|_{\mathbf{z}^*} = rs^*[-1] = -rs^*$
- $J_{43} = rs^* \frac{\partial \gamma_{rs}}{\partial L_F}\Big|_{\mathbf{z}^*} = rs^*\left[\eta\left(\beta - \frac{\beta_2}{1-\beta}\right)\right] = \eta\Xi_1 rs^*$, where Ξ_1 is defined above.
- $J_{44} = rs^* \frac{\partial \gamma_{rs}}{\partial rs}\Big|_{\mathbf{z}^*} = rs^*[1] = rs^*$

With all these elements, we can write the Jacobian matrix $\mathbf{J}_{4\times4}$. Thus, the stability of the steady state is determined by the eigenvalues of the Jacobian matrix $\mathbf{J}_{4\times4}$, which are the roots m of the characteristic equation $\det(\mathbf{J}_{4\times4} - m\mathbf{I}_4) = 0$, where $\mathbf{I}_4$ is the 4×4 identity matrix,

$$\mathbf{J}_{4\times4} - m\mathbf{I}_4 = \begin{pmatrix} -\beta(1-\beta)yk^* - m & 0 & \eta\Xi_1 yk^* & 0 \\ (\sigma(1-\beta)^2 - 1)ck^* & ck^* - m & 0 & 0 \\ \beta L_F^* & -L_F^* & \eta\Xi_2 L_F^* - m & 0 \\ \beta rs^* & -rs^* & \eta\Xi_1 rs^* & rs^* - m \end{pmatrix}$$

The determinant can be found using a Laplace expansion along the fourth column. All entries in the fourth column are zero except for the bottom entry $(J_{44} - m) = (rs^* - m)$. This yields:

$$\det(\mathbf{J}_{4\times4} - m\mathbf{I}_4) = (rs^* - m) \cdot (-1)^{4+4} \det B(m)$$

where $B(m)$ is the 3×3 minor obtained by removing the fourth row and fourth column:

$$B(m) = \begin{pmatrix} -\beta(1-\beta)yk^* - m & 0 & \eta\Xi_1 yk^* \\ (\sigma(1-\beta)^2 - 1)ck^* & ck^* - m & 0 \\ \beta L_F^* & -L_F^* & \eta\Xi_2 L_F^* - m \end{pmatrix}$$

The determinant of $B(m)$ is calculated as:

$$\begin{aligned} \det B(m) &= \big[-\beta(1-\beta)yk^* - m\big] \begin{vmatrix} ck^* - m & 0 \\ -L_F^* & \eta\Xi_2 L_F^* - m \end{vmatrix} \\ &\quad + \eta\Xi_1 yk^* \begin{vmatrix} (\sigma(1-\beta)^2 - 1)ck^* & ck^* - m \\ \beta L_F^* & -L_F^* \end{vmatrix} \\ &= \big[-\beta(1-\beta)yk^* - m\big]\big((ck^* - m)(\eta\Xi_2 L_F^* - m)\big) \\ &\quad + \eta\Xi_1 yk^*\big((\sigma(1-\beta)^2 - 1)ck^*(-L_F^*) - (ck^* - m)(\beta L_F^*)\big) \\ &= \big[-\beta(1-\beta)yk^* - m\big](ck^* - m)(\eta\Xi_2 L_F^* - m) \\ &\quad - \eta\Xi_1 yk^* L_F^*\big((\sigma(1-\beta)^2 - 1 + \beta)ck^* - \beta m\big) \end{aligned}$$

Expanding this expression gives a cubic polynomial in m:

$$\det B(m) = -m^3 + p_1 m^2 + p_2 m + p_3$$

Let $A_J = -\beta(1-\beta)yk^*$, $C_J = ck^*$, and $E_J = \eta\Xi_2 L_F^*$. The coefficients are:

$$p_1 = A_J + C_J + E_J = -\beta(1-\beta)yk^* + ck^* + \eta\Xi_2 L_F^* \tag{B.45a}$$

$$\begin{aligned} p_2 &= -(A_J C_J + A_J E_J + C_J E_J) + \eta\Xi_1 yk^* L_F^* \beta \\ &= \beta(1-\beta)yk^*(ck^* + \eta\Xi_2 L_F^*) - ck^*\eta\Xi_2 L_F^* + \beta\eta\Xi_1 yk^* L_F^* \end{aligned} \tag{B.45b}$$

$$\begin{aligned} p_3 &= A_J C_J E_J - \eta\Xi_1 yk^* L_F^*(\sigma(1-\beta)^2 - 1 + \beta)ck^* \\ &= -\beta(1-\beta)yk^* ck^* \eta\Xi_2 L_F^* - \eta\Xi_1 yk^* L_F^*(\sigma(1-\beta)^2 - 1 + \beta)ck^* \end{aligned} \tag{B.45c}$$

The characteristic polynomial for the economic core is therefore:

$$\det(\mathbf{J}_{4\times 4} - m\mathbf{I}_4) = (rs^* - m)(-m^3 + p_1 m^2 + p_2 m + p_3)$$

The four eigenvalues of the economic system are $m_4 = rs^*$ and the three roots of the cubic equation $m^3 - p_1 m^2 - p_2 m - p_3 = 0$.

C Euler Equation, Interest Rates, Overall Impatience and Evolution of Variables.

C.1 Theory and Simulation of the Effective Rate of Time Preference $\lambda(\sigma, r)$, and the Growth Rate $\gamma(\sigma, r)$

Here we study the relationship between the consumption growth rate $\gamma(\sigma, r)$ from the Euler equation and the interest rate r. We proceed in three steps: first, we formalize the effective discount rate $\lambda(\sigma, r)$ as a statistical expectation; second, we derive the sensitivity of growth to interest rates; and third, we analyze the geometric properties (existence, intercepts, and curvature) validated by numerical simulation.

C.1.1 Probabilistic Characterization of Time Preference

The effective rate of time preference, $\lambda(\sigma, r)$, is defined as the weighted average of the instantaneous discount rates $\rho(j)$:

$$\lambda(\sigma, r) \equiv \frac{\int_0^\infty \rho(j) \left[\theta(j)^\sigma e^{-(1-\sigma)rj}\right] dj}{\int_0^\infty \left[\theta(j)^\sigma e^{-(1-\sigma)rj}\right] dj}.$$

To analyze the behavior of this function rigorously, it is convenient to define the probability measure induced by the weighting kernel.

Definition 1 (Induced Probability Measure)**.** *Let the* ***weighting kernel*** $W(j; \sigma, r)$ *be defined as:*

$$W(j; \sigma, r) \equiv \theta(j)^\sigma e^{-(1-\sigma)rj}. \tag{C.1}$$

Let the ***partition function*** *(or total mass)* $Z(\sigma, r)$ *be the integral of the kernel:*

$$Z(\sigma, r) \equiv \int_0^\infty W(j; \sigma, r) dj. \tag{C.2}$$

Provided $Z < \infty$*, we define the probability density function* $f(j; \sigma, r)$ *as the normalized kernel:*

$$f(j; \sigma, r) \equiv \frac{W(j; \sigma, r)}{Z(\sigma, r)}. \tag{C.3}$$

Note that this behaves like a probability, because if we integrate it, we get 1, $\int_0^\infty f(j, r) dj = \frac{1}{Z(r)} \int_0^\infty W(j, r) dj = \frac{Z(r)}{Z(r)} = 1$. This allows us to interpret the effective discount rate $\lambda(\sigma, r)$ as the **expected value** of $\rho(j)$ under this induced measure:

$$\lambda(\sigma, r) = \mathbb{E}_r[\rho(j)] \equiv \int_0^\infty \rho(j) f(j; \sigma, r) dj. \tag{C.4}$$

C.1.2 Sensitivity Analysis and the Covariance Identity

We want to study the slope of the growth rate with respect to the interest rate, $\frac{\partial \gamma(\sigma,r)}{\partial r}$. From the Euler equation $\gamma = \sigma(r - \lambda(\sigma, r))$, this requires determining $\frac{\partial \lambda(\sigma,r)}{\partial r}$.

Applying the quotient rule to the definition $\lambda = U(r)/Z(r)$ (where $U(r)$ is the unnormalized integral $\int \rho W dj$):

$$\frac{\partial \lambda}{\partial r} = \frac{U'(r)}{Z(r)} - \frac{U(r)}{Z(r)} \frac{Z'(r)}{Z(r)}. \tag{C.5}$$

$$\frac{\partial \lambda}{\partial r} = \frac{U'(r)}{Z(r)} - \frac{U(r)}{Z(r)} \frac{Z'(r)}{Z(r)}. \tag{C.6}$$

First, we compute the derivative of the weighting function (or kernel) $W(j, r)$ with respect to r:

$$\frac{\partial W(j,r)}{\partial r} = \frac{\partial}{\partial r} \left(\theta(j)^{\sigma} e^{-(1-\sigma)rj} \right) = \left[-(1-\sigma)j \right] W(j,r). \tag{C.7}$$

The term Z'/Z becomes

$$\begin{aligned} \frac{Z'(r)}{Z(r)} &= \frac{1}{Z(r)} \int_0^{\infty} \frac{\partial W}{\partial r} dj \\ &= \int_0^{\infty} \left[-(1-\sigma)j \right] \frac{W(j,r)}{Z(r)} dj \\ &= \int_0^{\infty} \left[-(1-\sigma)j \right] f(j,r) dj \\ &= \mathbb{E}_r \left[-(1-\sigma)j \right]. \end{aligned} \tag{C.8}$$

The term U'/Z is then

$$\begin{aligned} \frac{U'(r)}{Z(r)} &= \frac{1}{Z(r)} \int_0^{\infty} \rho(j) \frac{\partial W}{\partial r} dj \\ &= \int_0^{\infty} \rho(j) \left[-(1-\sigma)j \right] f(j;r) dj \\ &= \mathbb{E}_r \left[\rho(j) \cdot \left(-(1-\sigma)j \right) \right]. \end{aligned} \tag{C.9}$$

Substituting these expectations back into (C.6):

$$\frac{\partial \lambda}{\partial r} = \mathbb{E}_r \left[\rho(j) \cdot \left(-(1-\sigma)j \right) \right] - \mathbb{E}_r[\rho(j)] \cdot \mathbb{E}_r \left[-(1-\sigma)j \right]. \tag{C.10}$$

Then, it is easy to see that using the definition of statistical covariance: $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$, we get the following result:

Lemma 5 (Sensitivity of the Discount Rate)**.** *The sensitivity of the effective discount rate to the interest rate is proportional to the covariance between the instantaneous discount rate $\rho(j)$ and the time horizon j:*

$$\frac{\partial \lambda}{\partial r} = -(1-\sigma) \cdot Cov_r\big(\rho(j),\, j\big). \tag{C.11}$$

Proposition 5 (The Growth Slope). *Under present bias, $\rho'(j) < 0$, implying $Cov(\rho, j) < 0$. The sensitivity of the growth rate is:*

$$\frac{\partial \gamma(\sigma, r)}{\partial r} = \sigma \left[1 + (1-\sigma) Cov_r(\rho(j), j)\right]. \tag{C.12}$$

1. *If $\sigma < 1$ (Inelastic): The growth response is dampened ($0 < \partial\gamma/\partial r < \sigma$).*

2. *If $\sigma > 1$ (Elastic): The growth response is amplified ($\partial\gamma/\partial r > \sigma$), given the conditions that the integrals in λ converge, and the same boundary restrictions on σ that appear in the text apply.*

Figure C.2 shows the relationship, first between $\lambda(\sigma, r)$ and r as capture by eq (C.11) and second, between the growth rate $\gamma(\sigma, r)$ and the interest rate r capture by eq (C.12). This is the numerical simulation of our model, showing the intuition of Propositions C.11 and 5.

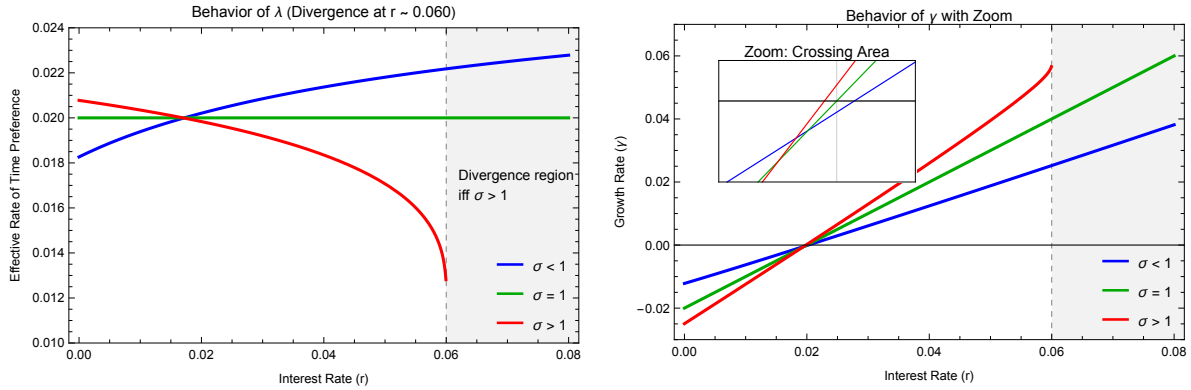


Figure C.3: Relationship between Growth Rate $\gamma(\sigma, r)$ and Interest Rate r.

We now also show the existence conditions (integrability) and the curvature seen in the simulated model.

C.1.3 Integrability and The Divergence Region

For the effective discount rate λ to be well-defined, the partition function $Z(\sigma, r)$ must converge. We analyze the asymptotic behavior of the integrand as $j \to \infty$. Let $\rho_\infty = \lim_{j\to\infty} \rho(j)$, which is greater or equal to zero. The integrand behaves as (see also Online Appendix A.1):

$$W(j; \sigma, r) \approx \exp\left\{-\left[\sigma\rho_\infty + (1-\sigma)r\right] j\right\}.$$

Convergence requires the effective decay rate to be positive:

$$\sigma\rho_\infty + (1-\sigma)r > 0. \tag{C.13}$$

- **Case $\sigma < 1$:** Since $(1-\sigma) > 0$, the condition holds for all $r \geq 0$ (assuming $\rho_\infty > 0$). The growth schedule is defined everywhere.

- **Case $\sigma > 1$:** Since $(1-\sigma) < 0$, the condition is violated if r is too large. There exists a critical interest rate:

$$r_{crit} \equiv \frac{\sigma\rho_\infty}{\sigma - 1}. \tag{C.14}$$

If $r \to r_{crit}$, the integral $Z(\sigma, r)$ diverges. In the main text we write is as a threshold for σ, such that $\sigma < \bar{\sigma} \equiv \frac{r}{r-\rho_\infty}$, which shows intuitively that σ cannot be arbitrarily large for a given r.

C.1.4 Analytical Results on the Intercepts of $\gamma(\sigma, r)$

As seen in Figure (C.3), the intercepts of the growth curves differ depending on the elasticity of substitution σ. It can be seen that the intercept of $\gamma(\sigma, r)$ is lower the higher the elasticity of substitution σ is. We now derive two key theoretical properties of the growth rate $\gamma(\sigma, r)$ that explain the shape of the curves.

Proposition 6 (Ordering of Growth Intercepts)**.** *Let $r = 0$. If the instantaneous discount rate $\rho(j)$ is strictly decreasing in j (when agents are present-biased), then the growth intercept $\gamma(\sigma, 0)$ is a strictly decreasing function of the elasticity of substitution σ. Consequently:*

$$\gamma_{\sigma>1}(0) < \gamma_{\sigma=1}(0) < \gamma_{\sigma<1}(0). \tag{C.15}$$

Proof. At $r = 0$, the growth rate is $\gamma(\sigma, 0) = -\sigma\lambda(\sigma, 0)$. We first determine the sign of $\frac{\partial\lambda(\sigma,0)}{\partial\sigma}$. The effective discount rate at $r = 0$ is the expected value of $\rho(j)$ weighted by $\theta(j)^\sigma$:

$$\lambda(\sigma, 0) = \frac{\int \rho(j)\theta(j)^\sigma dj}{\int \theta(j)^\sigma dj} = \mathbb{E}_0[\rho(j)].$$

Differentiating with respect to σ (similar to the derivation above), the derivative of the log-weight is $\frac{\partial}{\partial\sigma}(\sigma \ln\theta(j)) = \ln\theta(j)$. Thus, the derivative of the expectation is the covariance:

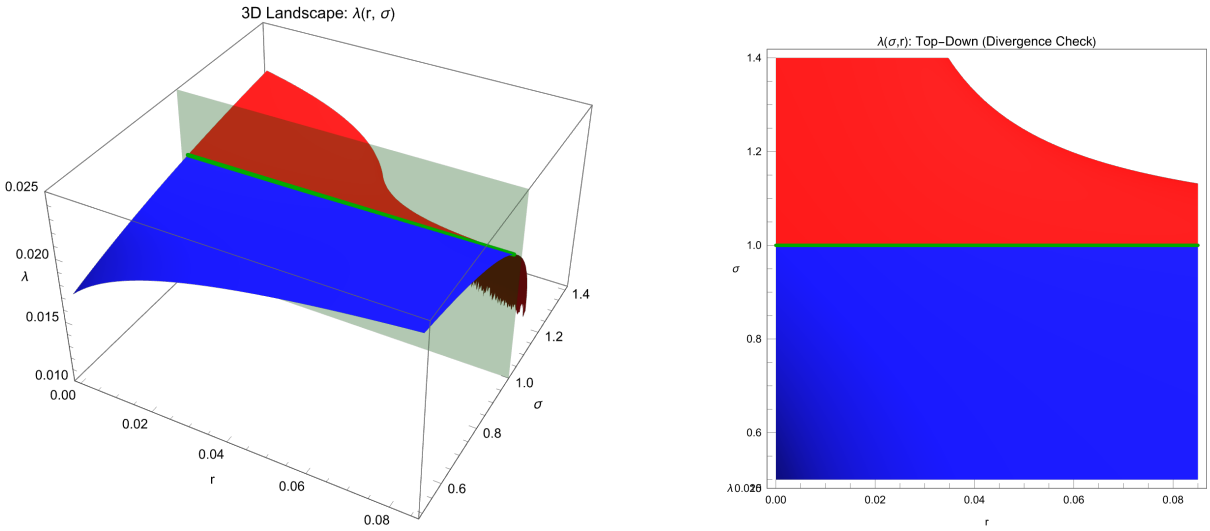
$$\frac{\partial\lambda(\sigma, 0)}{\partial\sigma} = \mathrm{Cov}_0\big(\rho(j), \ln\theta(j)\big).$$

We analyze the monotonicity of the two variables:

1. $\rho(j)$ is strictly decreasing in j (for present-biased agents).
2. $\ln\theta(j)$ is strictly decreasing in j (since $\theta(j)$ is the discount function, $\dot{\theta}(j) < 0$).

By Chebyshev's Sum Inequality (or the properties of covariance), the covariance of two similarly monotonic functions is *positive*. Thus, $\frac{\partial\lambda}{\partial\sigma} > 0$. Now consider $\gamma(\sigma, 0) = -\sigma\lambda(\sigma, 0)$. Since $\lambda > 0$ and is increasing in σ, and the pre-factor $-\sigma$ is decreasing, the total product is strictly decreasing. Thus, a higher σ results in a strictly more negative intercept. □

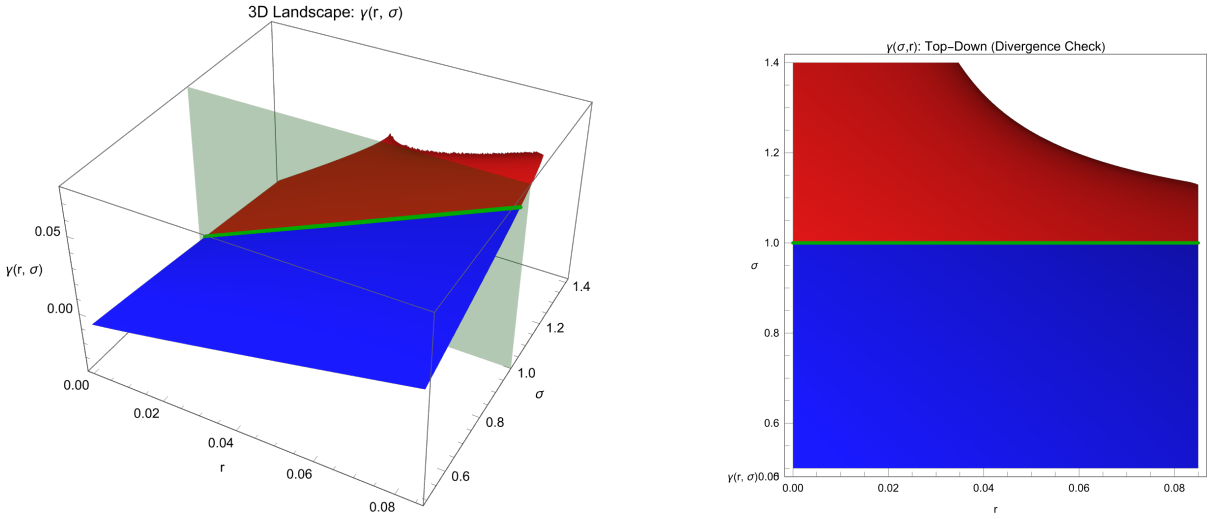
For further intuition, we show the 3D plot first for $\lambda(\sigma, r)$ in Figure C.4, and then $\gamma(\sigma, r)$ in Figure C.5. This shows all the theoretical properties derived above. Additionally, observe that the green line and "vertical wall" corresponds to the edge log case when $\sigma = 1$. Moreover, one can see that for $\sigma < 1$, the function is well-defined for all $r \geq 0$ (blue part of the function), while for $\sigma > 1$, there is a divergence region as explained above (red region). This visually and intuitively shows that σ cannot be arbitrarily large for a given r, as captured by the integrability condition above.



(a) Perspective view of $\lambda(\sigma, r)$

(b) Top-down view of $\lambda(\sigma, r)$

Figure C.4: Two views of the effective discount rate $\lambda(\sigma, r)$.



(a) Perspective view of $\gamma(\sigma, r)$

(b) Top-down view of $\gamma(\sigma, r)$

Figure C.5: Two views of the growth rate $\gamma(\sigma, r)$.

Parameters	Description	Value
$r_{Exp}^{Ces\ \sigma<1}$	Equilibrium Interest rate, CES ($\sigma < 1$) & Exp	0.0391777
r_{Exp}^{log}	Equilibrium Interest rate, Log & Exp	0.0379999
$r_{Exp}^{Ces\ \sigma>1}$	Equilibrium Interest rate, CES ($\sigma > 1$) & Exp	0.0373603
$r_{Tsoukis}^{ces,\ \sigma<1}$	Equilibrium Interest rate, CES ($\sigma < 1$) & Tsoukis	0.0393529
$r_{Tsoukis}^{log}$	Equilibrium Interest rate, Log & Tsoukis	0.0379999
$r_{Tsoukis}^{ces,\ \sigma>1}$	Equilibrium Interest rate, CES ($\sigma > 1$) & Tsoukis	0.0371896
$r_{ConvexExp}^{ces,\ \sigma<1}$	Equilibrium Interest rate, CES ($\sigma < 1$) & ConvexExp	0.0394499
$r_{ConvexExp}^{log}$	Equilibrium Interest rate, Log & ConvexExp	0.0379999
$r_{ConvexExp}^{ces,\ \sigma>1}$	Equilibrium Interest rate, CES ($\sigma > 1$) & ConvexExp	0.0371190

Table C.2: Equilibrium Interest Rates

C.2 Equilibrium Interest Rates

Table C.2 shows the equilibrium interest rates generated by the model. One can see that with log utility ($\sigma = 1$), the interest rates are the same across all discount functions. However, when we move to $\sigma \neq 1$, the interest rates differ depending on weather the agents in the economy are present-biased or not. For $\sigma < 1$, the equilibrium interest rates are higher when agents are present-biased, while for $\sigma > 1$, the equilibrium interest rates are lower when agents are present-biased.

C.3 Structural Parameters Used in the Quantitative Analysis.

Table C.3 summarizes the structural parameter values used in the quantitative analysis.

C.4 Controling for overall impatience in the discount functions

We work with different discount functions to see their effects on how agents make decisions and the impact on economic growth, how agents extract the exhaustible resource and many different and interesting aspects of our model. First, we show the discount function (19) which appears in Tsoukis et al. (2017) and Cabo et al. (2020a). Thus, we need to find the right parameters for the non-constant discounting as in expression (21), where the discounted infinite stream of both discount functions gives the same present value. This last expression will be used to compare all the different discount functions

Table C.3: Structural Parameter Values

Parameters	Description	Value
β_1	Output Elasticity of Final Labor	0.66
β_2	Output Elasticity of Resource	0.04
$1-\beta_1-\beta_2$	Output Elasticity of Intermediate Machines	0.3
σ	Intertemporal elasticity of substitution	$\left\{\frac{1}{1.5}, 1, 1.2\right\}$
$\hat{\rho}$	Instantaneous Discount Rate for the Exponential	0.02
η	Parameter of the Innovation Possibilities Frontier	0.142163
ψ	From disutility of polution	0.1
b	Curvature of disutility of polution	0.1

Note: Baseline parameter values used for the simulation. The elasticity of intertemporal substitution σ takes three distinct values to illustrate the interaction with time inconsistency.

to be equivalent to the exponential discount.

- For the discount function used in Tsoukis et al. (2017), this implies the following relationship

$$\int_0^\infty e^{-\hat{\rho}j}dj \stackrel{!}{=} \int_0^\infty (1+\delta j)^{-\varphi/\delta}e^{-\rho j}dj, \quad \Longleftrightarrow \quad \hat{\rho} = \frac{e^{-\frac{\rho}{\delta}}\delta^{\frac{\varphi}{\delta}}\rho^{1-\frac{\varphi}{\delta}}}{\Gamma\left(1-\frac{\varphi}{\delta}, \frac{\rho}{\delta}\right)}, \tag{C.16}$$

where $\Gamma(a,b) = \int_b^\infty x^{a-1}e^{-x}dx$ is the incomplete gamma function. The intuition behind this expression is to capture a constant rate of time preferences $\hat{\rho}$ such that it gives the same overall level of impatience with the exponential discount (time-consistent) as the overall level of impatience under a non-constant discounting (see e.g., Barro (1999), Strulik (2015) or Cabo et al. (2020a)).

Following the Stern Review Stern, 2007, Box 6.3, p. 184, we set the parameter $\rho = 1\%$ in the Tsoukis discount function. For the corresponding parameters φ and δ, we follow the approach in Dugan and Trimborn (2020), where the parameters of the discount function are driven by the data in Weitzman (2001), in which the author studies a numerical example with data from a survey based on the opinions of 2,160 economists (see Table 2 in the mentioned work to see the different discount rate for different distant times).[43] With that data, Dugan and Trimborn (2020) obtain what would be in our work a value of $\delta = 0.021$. Together with $\rho = 0.01$, one should solve for φ in the equation (C.16)

[43]The author asked the question "Taking all relevant considerations into account, what real interest rate do you think should be used to discount over time the (expected) benefits and (expected) costs of projects being proposed to mitigate the possible effects of global climate change?".

(where $\hat{\rho} = 0.02$), obtaining a value of $\varphi = 0.01786073$. Observe that without setting a parameter, there would be a continuum of solutions for the couple of parameters. All parameters related to the discounting functions are provided in Table C.4. Thus, as highlighted previously, it is easy to see that both integrals report the same discounted value

$$\int_0^\infty e^{-\hat{\rho}j} dj = \int_0^\infty (1+\delta j)^{-\varphi/\delta} e^{-\rho j} dj,$$
$$\Leftrightarrow \frac{1}{\hat{\rho}} = e^{\rho/\delta} \delta^{-\frac{\varphi}{\delta}} \rho^{\frac{\varphi}{\delta}-1} \Gamma\left(1 - \frac{\varphi}{\delta}, \frac{\rho}{\delta}\right) = 50 \text{ (with the given parameters).}$$

- For the convex linear combination of two exponentials discount function (20), we have the following relationship

$$\int_0^\infty e^{-\hat{\rho}j} dj \overset{!}{=} \int_0^\infty z e^{-\rho_1 j} + (1-z) e^{-\rho_2 j} dj, \quad \Longleftrightarrow \quad \hat{\rho} = \frac{\rho_1 \rho_2}{\rho_1 - z\rho_1 + z\rho_2}, \tag{C.17}$$

where we use the same parameters as in Mañó-Cabello et al. (2021), with $z = 0.5$, i.e., an equal weight for the long run and the short run impatience. The short run impatience is governed by $\rho_1 = 0.05$ and the long run impatience by $\rho_2 = 0.0125$. As before, given $\hat{\rho}$ from the exponential discount, the weight z and ρ_1, one should look for the parameter governing the short run impatience ρ_2 to give the same stream of the integral with an exponential discount, as it is shown in (C.17). One can easily check that both integrals give the same value using the parameters in Table C.4.

Table C.4: Parameter Values for Different Discount Functions $\theta(j)$

	$e^{-\hat{\rho}}$	$e^{-\rho j}(1+\delta j)^{-\varphi/\delta}$			$ze^{-\rho_1 j} + (1-z)e^{-\rho_2 j}$		
Parameters	$\hat{\rho}$	ρ	δ	φ	z	ρ_1	ρ_2
Values	0.02	0.01	0.021	0.01786073	0.5	0.05	0.0125

Note: Parameter values for the three discount functions used in the quantitative analysis: Exponential, Tsoukis, and Convex Exponential.

C.5 Evolution of the Extraction Rate $R(s)/S(s)$ under different discount functions.

Figure C.6 shows the evolution of the extraction rate $R(s)/S(s)$ under different discount functions.

C.6 Time evolution of Pollution under different regimes

The long-run behavior of pollution in the model is governed by the pollution intensity parameters μ_1 (from resource extraction) and μ_2 (from final good production). To explore all thre possible outcomes, we consider different scenarios of pollution intensity parameters (μ_1, μ_2).

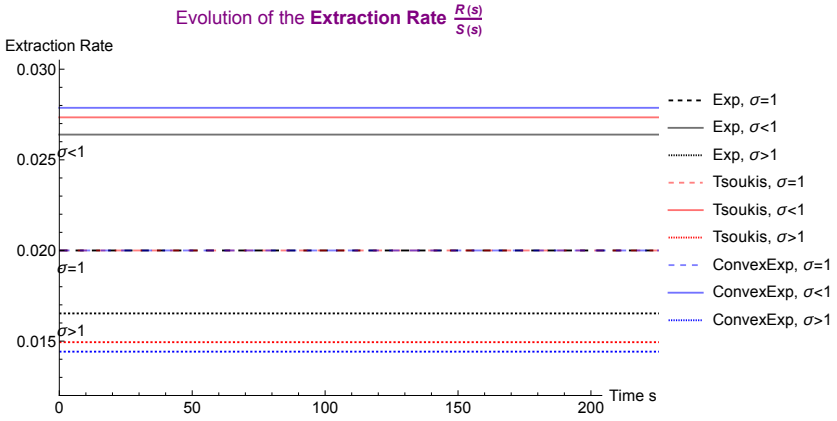


Figure C.6: Evolution of the Extraction Rate $R(s)/S(s)$ under different discount functions.

From eq (40), the steady-state growth rate of pollution is:

$$\gamma_P = \mu_2\gamma_N + \left(\mu_1 + \mu_2\frac{\beta_2}{\beta_1+\beta_2}\right)\gamma_R$$

The Business-as-Usual (BAU) scenario, where $\gamma_P = 0$, requires a specific balance between these parameters. Rearranging the equation for this condition gives the required value of μ_1 as a function of μ_2:

$$\mu_{1,\gamma_P=0} = -\mu_2\left(\frac{\gamma_N}{\gamma_R} + \frac{\beta_2}{\beta_1+\beta_2}\right)$$

Since the steady-state growth rates γ_N and γ_R depend on the household's EIS (σ) and their discount function, the value of μ_1 required to achieve the BAU target is specific to each preference specification. For our simulations, we fix $\mu_2 = 0.005$ for the BAU case and then calculate the corresponding μ_1 for each scenario.

For the declining pollution (UN) and rising pollution (CHKT) scenarios, we simply choose parameter pairs that ensure $\gamma_P < 0$ and $\gamma_P > 0$, respectively, across all preference specifications. The full set of calibrated pollution parameters used for the simulations is reported in Table C.5.

Figures C.7a, C.7b and C.7c show the time evolution of pollution for the three different regimes of γ_P (negative, zero and positive), and observing how time-inconsistency affects its dynamics.

When pollution declines over time, under $\sigma < 1$, the economy populated by time-inconsistent agents pollutes more at the beggining than the economy with time-consistent agents (exponential discounting). The opposite happens when $\sigma > 1$. However, as time goes by, the story reverses, and the economy with time-inconsistent agents pollutes less than the economy with time-consistent agents when $\sigma < 1$, and more when $\sigma > 1$.

In the case of a zero growth rate of pollution, i.e., constnat pollution over time, the economy with time-inconsistent households pollutes more if and only if $\sigma < 1$, and less when $\sigma > 1$.

Finally, when pollution increases over time, the economy with time-inconsistent agents pollutes less than the economy with time-consistent agents when $\sigma < 1$, and more when $\sigma > 1$.

Table C.5: Pollution Intensities μ_i Used in Different Model Versions

Parameters	Description	Value
$\mu_{1,\gamma_P^\theta<0}$	Final Production Intensity in Resource Extraction	0.9
$\mu_{1,\gamma_P^\theta=0}^{\theta,\log}$	Final Production Intensity in Pollution	0.00449998
$\mu_{1,\gamma_P^\theta=0}^{\text{Exp,Ces},\sigma<1}$	Final Production Intensity in Pollution	0.00242211
$\mu_{1,\gamma_P^\theta=0}^{\text{Exp,Ces},\sigma>1}$	Final Production Intensity in Pollution	0.00630215
$\mu_{1,\gamma_P^\theta=0}^{\text{Ts,Ces},\sigma<1}$	Final Production Intensity in Pollution	0.00219603
$\mu_{1,\gamma_P^\theta=0}^{\text{Ts,Ces},\sigma>1}$	Final Production Intensity in Pollution	0.0069184
$\mu_{1,\gamma_P^\theta=0}^{\text{ConEx}}$	Final Production Intensity in Pollution	0.00449998
$\mu_{1,\gamma_P^\theta=0}^{\text{ConEx,Ces},\sigma<1}$	Final Production Intensity in Pollution	0.00207745
$\mu_{1,\gamma_P^\theta=0}^{\text{ConEx,Ces},\sigma>1}$	Final Production Intensity in Pollution	0.0071953
$\mu_{1,\gamma_P^\theta>0}$	Final Production Intensity in Pollution	0.005
$\mu_{2,\gamma_P^\theta<0}$	Resource Extraction Intensity in Pollution	0.05
$\mu_{2,\gamma_P^\theta=0}$	Resource Extraction Intensity in Pollution	0.005
$\mu_{2,\gamma_P^\theta>0}$	Resource Extraction Intensity in Pollution	0.5

Table C.5 shows the values of the parameters μ_1 and μ_2 used in the different regimes of pollution growth rate γ_P.

C.7 Initial values of extraction, production, pollution and growth rate of pollution

Table C.6 shows the initial values for extraction ($R(0)$), production ($Y(0)$), and pollution ($P(0)$), as well as the growth rate of pollution (γ_P) for time-consistent agents (Exponential) vs present bias agents. The values are reported for different elasticity of substitution (σ) scenarios.

C.8 Time evolution of main economic variables under different regimes

Figure C.8 summarizes the overall economic performance by showing the evolution for Capital (K), Output (Y), Blueprints/Ideas (N), and Consumption (c). The simulations show graphically oru central result. When the EIS is high ($\sigma > 1$), the economy populated by time-inconsistent agents consistently outperforms the time-consistent benchmark, achieving higher long-run levels across all key macroeconomic aggregates. In contrast, when the EIS is low ($\sigma < 1$), the conventional intuition holds: the

Table C.6: Values of initial extraction, production, pollution and growth rate of pollution.

		CES $\sigma < 1$			Log			CES $\sigma > 1$		
		$\gamma_P < 0$	$\gamma_P = 0$	$\gamma_P > 0$	$\gamma_P < 0$	$\gamma_P = 0$	$\gamma_P > 0$	$\gamma_P < 0$	$\gamma_P = 0$	$\gamma_P > 0$
Exponential	$R(0)$	0.0132	0.0132	0.0132	0.0100	0.0100	0.0100	0.0083	0.0083	0.0083
	$Y(0)$	115.440	115.440	115.440	111.000	111.000	111.000	108.365	108.365	108.365
	$P(0)$	0.026	1.013	10.514	0.020	1.003	10.296	0.017	0.993	10.163
	γ_P	-2.311%	0%	0.626%	-1.710%	0%	0.890%	-1.383%	0%	1.033%
Tsoukis	$R(0)$	0.0137	0.0137	0.0137	0.0100	0.0100	0.0100	0.0078	0.0078	0.0078
	$Y(0)$	116.067	116.067	116.067	111.000	111.000	111.000	107.625	107.625	107.625
	$P(0)$	0.027	1.014	10.545	0.020	1.003	10.296	0.016	0.990	10.126
	γ_P	-2.401%	0%	0.587%	-1.710%	0%	0.890%	-1.296%	0%	1.072%
ConvexExp	$R(0)$	0.0139	0.0139	0.0139	0.0100	0.0100	0.0100	0.0076	0.0076	0.0076
	$Y(0)$	116.411	116.411	116.411	111.000	111.000	111.000	107.313	107.313	107.313
	$P(0)$	0.027	1.015	10.561	0.020	1.003	10.296	0.016	0.988	10.110
	γ_P	-2.450%	0%	0.565%	-1.710%	0%	0.890%	-1.260%	0%	1.087%

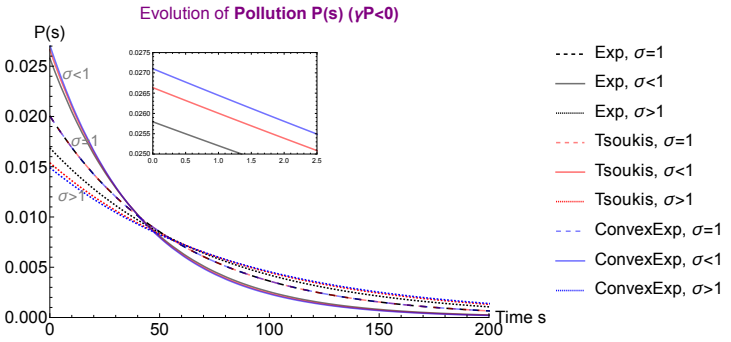
Note: This Table shows the initial values for extraction ($R(0)$), production ($Y(0)$), and pollution ($P(0)$), as well as the growth rate of pollution (γ_P) for time-consisntent agents (Exp) vs present bias agents. Columns correspond to different elasticity of substitution (σ) scenarios.

time-consistent economy dominated by rational exponential discounters achieves superior outcomes.

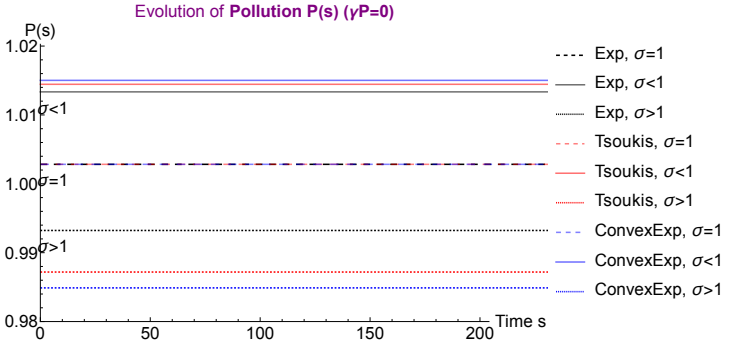
C.9 Visualizing System Dynamics. The Saddle-Point 3D Vector Field

To provide a geometric intuition for the system's saddle-point dynamics, Figure C.9 plots the 3D vector field for the core economic variables. See Table 3 for the corresponding eigenvalues and steady-state values. The flow lines illustrate that for any initial condition not on the one-dimensional stable manifold, the economy's trajectory diverges from the steady state. Only paths starting exactly on this manifold will converge. Note that most arrows point away from the steady state, showing the inherent instability of the system outside the stable manifold.

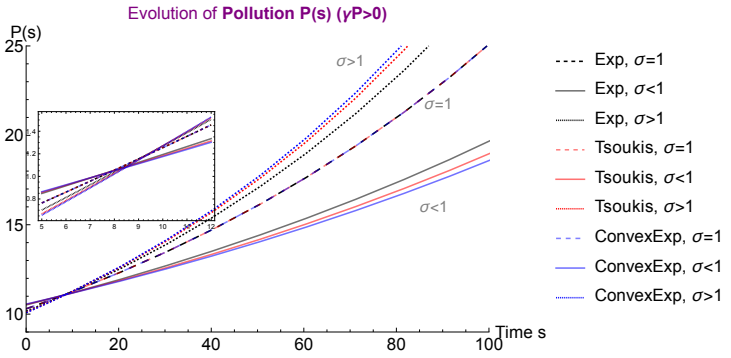
The figure also clearly shows the effect of present bias. For the case of low EIS ($\sigma < 1$), the steady states for the three different discount functions are plotted. The distinct locations of the equilibria for the Exponential (time-consistent) and the Tsoukis/ConvexExp (time-inconsistent) specifications provide a clear graphical representation of how present bias alters the long-run equilibrium of the economy when utility is not logarithmic.



(a) Evolution of pollution when $\gamma_P < 0$ with all different discount functions

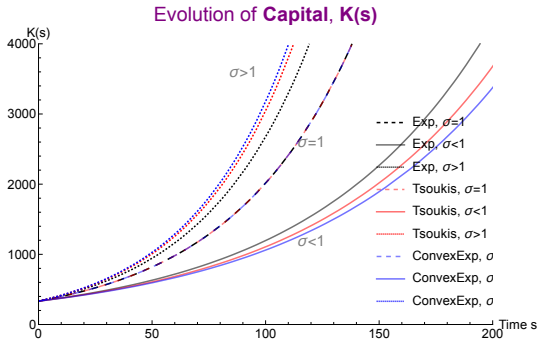


(b) Evolution of pollution when $\gamma_P = 0$ with all different discount functions

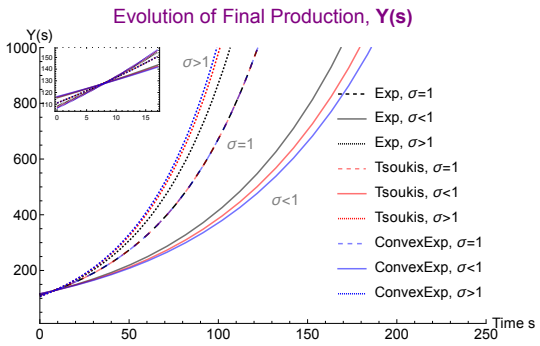


(c) Evolution of pollution when $\gamma_P > 0$ with all different discount functions

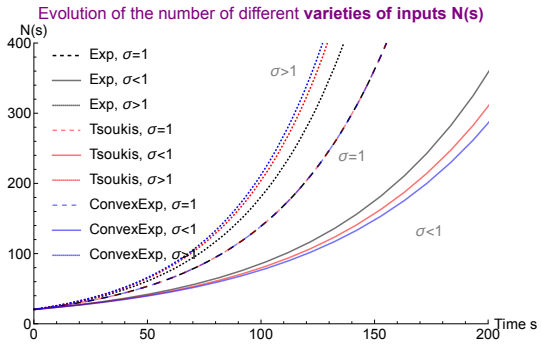
Figure C.7: Evolution of pollution across γ_P regimes with all discount functions.



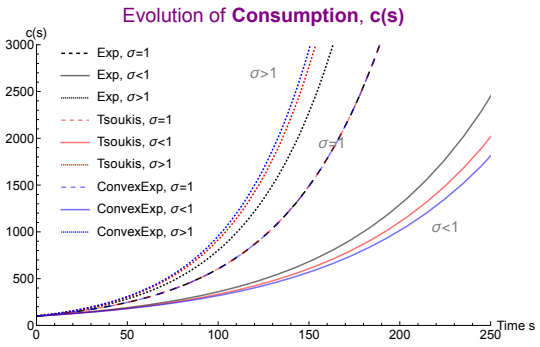
(a) Evolution of Capital $K(s)$.



(b) Evolution of Final Production $Y(s)$.



(c) Evolution of new Blueprints/ideas $N(s)$.



(d) Evolution of Consumption $c(s)$.

Figure C.8: Evolution of relevant variables on the model under different Discount Functions.

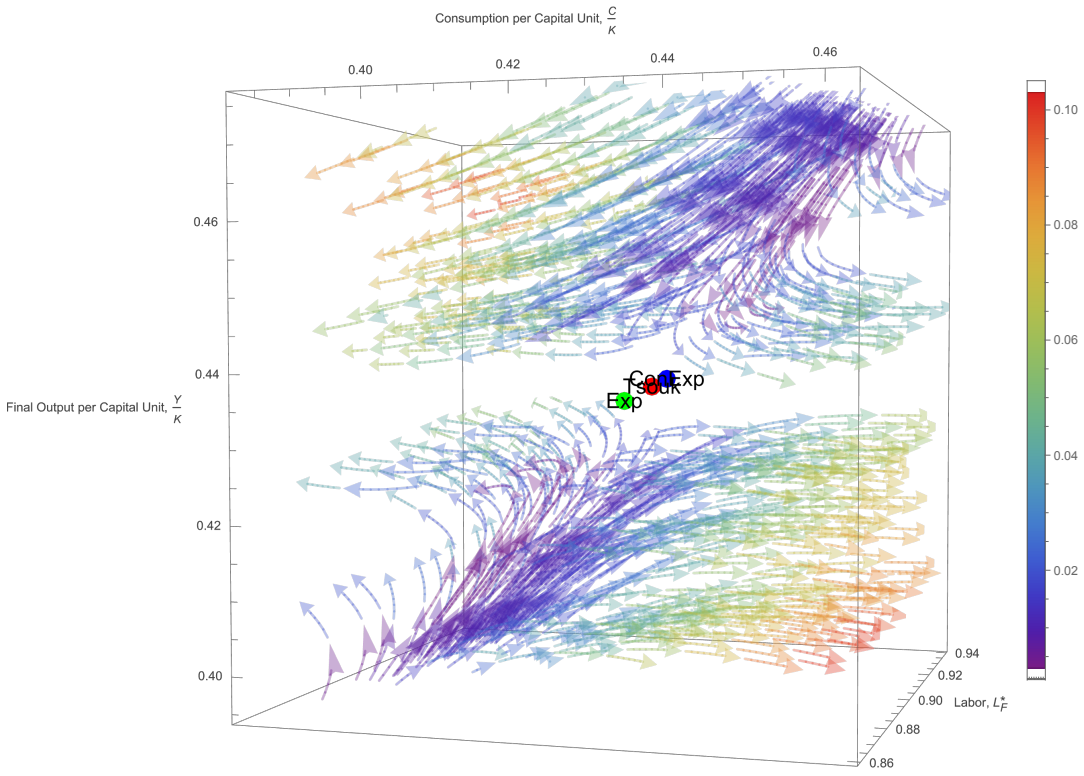


Figure C.9: 3D Vector Field Illustrating Saddle-Point Dynamics and Steady States $\sigma < 1$.

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